ROBBINS'S NEW PLANE GEOMETRY

BY

EDWARD RUTLEDGE ROBBINS, A.B.

FORMERLY OF LAWRENCEVILLE SCHOOL

AMERICAN BOOK COMPANY

NEW YORK CINCINNATI CHICAGO

ROBBINS'S NEW PLANE GEOMETRY.
W. P. I

FOR THOSE WHOSE PRIVILEGE

IT MAY BE TO ACQUIRE A KNOWLEDGE OF

GEOMETRY

THIS VOLUME HAS BEEN WRITTEN

AND TO THE BOYS AND GIRLS WHO LEARN THE ANCIENT SCIENCE

FROM THESE PAGES, AND WHO ESTEEM THE POWER

OF CORRECT REASONING THE MORE

BECAUSE OF THE LOGIC OF

PURE GEOMETRY

THIS VOLUME IS DEDICATED

PREFACE

THIS New Plane Geometry is not only the outgrowth of the author's long experience in teaching geometry, but has profited further by suggestions from teachers who have used Robbins's "Plane Geometry" and by many of the recommendations of the "National Committee of Fifteen." While many new and valuable features have been added in the reconstruction, yet all the characteristics that met with widespread favor in the old book have been retained.

Among the features of the book that make it sound and teachable may be mentioned the following:

1. The book has been written for the pupil. The objects sought in the study of Geometry are (1) to train the mind to accept only those statements as truth for which convincing reasons can be provided, and (2) to cultivate a foresight that will appreciate both the purpose in making a statement and the process of reasoning by which the ultimate truth is established. Thus, the study of this formal science should develop in the pupil the ability to pursue argument coherently, and to establish geometric truths in logical order. To meet the requirements of the various degrees of intellectual capacity and maturity in every class, the reason for every statement is not printed in full but is indicated by a reference. The pupil who knows the reason need not consult the paragraph cited; while the pupil who does not know it may learn it by the reference. It is obvious that the greater progress an individual makes in assimilating the subject and in entering into its spirit, the less need there will be for the printed reference.

2. Every effort has been made to stimulate the mental activity of the pupil. To compel a young student, however, to supply his

own demonstrations frequently proves unprofitable as well as arduous, and engenders in the learner a distaste for a study in which he might otherwise take delight. This text does not aim to produce accomplished geometricians at the completion of the first book, but to aid the learner in his progress throughout the volume, wherever experience has shown that he is likely to require assistance. It is designed, under good instruction, to develop a clear conception of the geometric idea, and to produce at the end of the course a rational individual and a friend of this particular science.

3. The theorems and their demonstrations — the real subject-matter of Geometry — are introduced as early in the study as possible.

4. The simple fundamental truths are explained instead of being formally demonstrated.

5. The original exercises are distinguished by their abundance, their practical bearings upon the affairs of life, their careful gradation and classification, and their independence. Every exercise can be solved or demonstrated without the use of any other exercise. Only the truths in the numbered paragraphs are necessary in working originals.

6. The exercises are introduced as near as practicable to the theorems to which they apply.

7. Emphasis is given to the discussion of original constructions.

8. The summaries will be found a valuable aid in reviews.

9. The historical notes give the pupil a knowledge of the development of the science of geometry and add interest to the study.

10. The attractive open page will appeal alike to pupils and to teachers.

The author sincerely desires to extend his thanks to those friends and fellow teachers who, by suggestion and encouragement, have inspired him in the preparation of these pages.

EDWARD R. ROBBINS.

CONTENTS

INTRODUCTION

BOOK I. ANGLES, LINES, RECTILINEAR FIGURES

BOOK II. THE CIRCLE

PLANE GEOMETRY

INTRODUCTION

1. **Geometry** is a science which treats of the measurement of magnitudes.

2. A **point** is that which has position but not magnitude.

3. A **line** is that which has length but no other magnitude.

4. A **straight line** is a line which is determined (fixed in position) by *any* two of its points. That is, two lines that coincide entirely, if they coincide at *any* two points, are straight lines.

5. A **rectilinear figure** is a figure containing straight lines and no others.

6. A **surface** is that which has length and breadth but no other magnitude.

7. A **plane** is a surface in which if any two points are taken, the straight line connecting them lies wholly in that surface.

8. **Plane Geometry** is a science which treats of the properties of magnitudes in a plane.

9. A **solid** is that which has length, breadth, and thickness. A solid is that which occupies space.

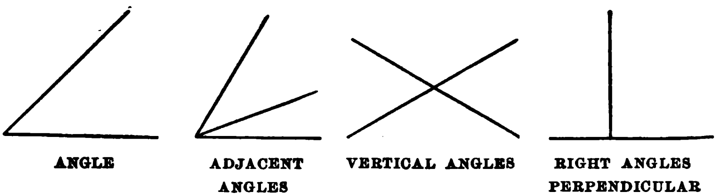
10. **Boundaries.** The boundaries (or boundary) of a solid are surfaces. The boundaries (or boundary) of a surface

are lines. The boundaries of a line are points. These boundaries can be no part of the things they limit. A surface is no part of a solid; a line is no part of a surface; a point is no part of a line.

11. Motion. If a point moves, its path is a line. Hence, if a point moves, it generates (describes or traces) a line; if a line moves (except upon itself), it generates a surface; if a surface moves (except upon itself), it generates a solid.

NOTE. Unless otherwise specified the word "line" means *straight line.*

ANGLES



12. A **plane angle** is the amount of divergence of two straight lines that meet. The lines are called the **sides** of the angle. The **vertex** of an angle is the point at which the lines meet.

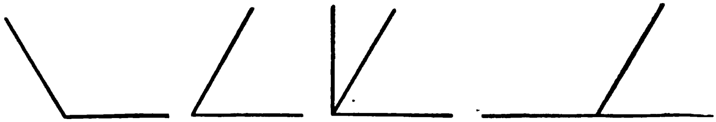
13. Adjacent angles are two angles that have the same vertex and a common side *between them.*

14. Vertical angles are two angles that have the same vertex, the sides of one being prolongations of the sides of the other.

15. If one straight line meets another and makes the adjacent angles equal, the angles are **right angles.**

16. One line is **perpendicular** to another if they meet at right angles. Either line is perpendicular to the other. The point at which the lines meet is the **foot** of the perpendicular. **Oblique lines** are lines that meet but are not perpendicular.

17. A **straight angle** is an angle whose sides lie in the same straight line, but extend in opposite directions from the vertex.



OBTUSE ANGLE — ACUTE ANGLE — COMPLEMENTARY ANGLES — SUPPLEMENTARY ANGLES

18. An **obtuse angle** is an angle that is greater than a right angle. An **acute angle** is an angle that is less than a right angle. An **oblique angle** is any angle that is not a right angle.

19. Two angles are **complementary** if their sum is equal to *one* right angle. Two angles are **supplementary** if their sum is equal to *two* right angles. Thus, the complement of an angle is the difference between one right angle and the given angle. The supplement of an angle is the difference between two right angles and the given angle.

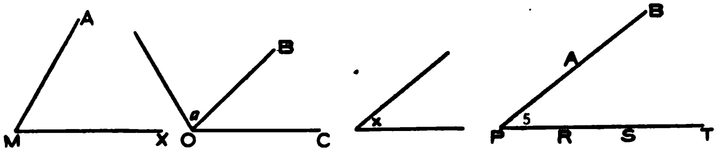
20. A **degree** is one ninetieth of a right angle. The degree is the familiar unit used in measuring angles. It is evident that there are 90° in a right angle; 180° in two right angles, or a straight angle; 360° in four right angles.

There are 60 minutes (60′) in one degree, and 60 seconds (60″) in one minute.

21. **Parallel lines** are straight lines that lie in the same plane and that never meet, however far they are extended in either direction.

22. Notation. A point is usually denoted by a capital letter, placed near it. A line is denoted by two capital letters, placed one at each end, or one at each of two of its points. Its length is sometimes represented advantageously by a small letter written near it. Thus, the line AB; the line RS; the line m.

There are various ways of naming angles. Sometimes three capital letters are used, one on each side of the angle and one at the vertex; sometimes a small letter or a figure is placed within the angle. The symbol for angle is $\angle$.



$\angle AMX$ OR $\angle XMA$ OR $\angle M$ $\angle a$ AND $\angle BOC$, NOT $\angle O$ $\angle x$

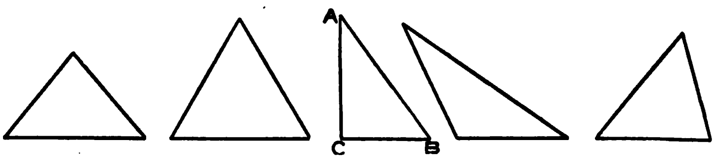
In naming an angle by the use of three letters, the vertex letter is always placed between the others. Thus the $\measuredangle$ above are $\angle AMX$ or $\angle XMA$, $\angle a$, $\angle BOC$, $\angle x$, $\angle APR$, $\angle APS$, $\angle BPR$, $\angle TPB$, $\angle 5$, etc.

In the above figure $\angle x = \angle 5$. The size of an angle depends on the amount of divergence between its sides, and not upon their length.

An angle is said to be **included** by its sides. An angle is **bisected** by a line drawn through the vertex and dividing the angle into two equal angles.

TRIANGLES

23. A **triangle** is a portion of a plane bounded by three straight lines. These lines are the **sides**. The **vertices** of a triangle are the three points at which the sides intersect. The **angles** of a triangle are the three angles at the three vertices. Each side of a triangle has two angles adjoining it. The symbol for triangle is $\triangle$.



ISOSCELES $\triangle$ EQUILATERAL $\triangle$ EQUIANGULAR $\triangle$ RIGHT $\triangle$ OBTUSE $\triangle$ ACUTE $\triangle$ SCALENE $\triangle$

The **base** of a triangle is the side on which the figure appears to stand. The **vertex** of a triangle is the vertex opposite the base. The **vertex angle** is the angle opposite the base.

24. Kinds of triangles:

A **scalene** triangle is a triangle **no two sides** of which are equal.
An **isosceles** triangle is a triangle **two sides** of which are equal.
An **equilateral** triangle is a triangle **all sides** of which are equal.
A **right** triangle is a triangle **one angle** of which is a right angle.
An **obtuse** triangle is a triangle **one angle** of which is an obtuse angle.
An **acute** triangle is a triangle **all angles** of which are acute angles.
An **equiangular** triangle is a triangle **all angles** of which are equal.

25. The **hypotenuse** of a right triangle is the side opposite the right angle. The sides forming the right angle are called **legs**.

CONGRUENCE

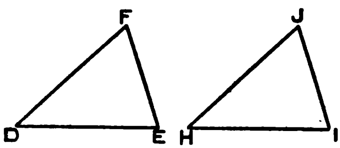
26. Two geometric figures are said to be **equal** if they have the same size or magnitude.

Two geometric figures are said to be **congruent** if, when one is superposed upon the other, they coincide in all respects.

The corresponding parts of congruent figures are equal, and are called **homologous parts**.

27. Homologous parts of congruent figures are equal.

If the triangles DEF and HIJ are congruent,
$\angle D$ is homologous to and = to $\angle H$;
DE is homologous to and = to HI;
$\angle E$ is homologous to and = to $\angle I$;
EF is homologous to and = to IJ.



NOTE. **Congruent figures** have the same shape as well as the same size, whereas **equal figures** do not necessarily have the same shape.

Ex. 1. What is the complement of an angle of 35°? 48°? 80°? 75° 50′? 8° 20′?

Ex. 2. What is the supplement of an angle of 100°? 50°? 148°? 121° 30′? 10° 40′?

28. A **curve** or **curved line,** is a line no part of which is straight.

A **circle** is a plane curve all points of which are equally distant from a point in the plane, called the **center.**

An **arc** is any part of a circle.

A **radius** is a straight line from the center to any point of the circle.

A **diameter** is a straight line containing the center and having its extremities in the circle.

The length of the circle is called the **circumference.**

29. Symbols. The usual symbols and abbreviations employed in geometry are the following:

+	plus.	⊙	circle.	Ax.	axiom.
−	minus.	Ⓢ	circles.	Hyp.	hypothesis.
=	equals, is equal to, equal.	∠	angle.	comp.	complementary.
		∡	angles.	supp.	supplementary.
≠	does not equal.	rt. ∠	right angle.	Const.	construction.
≅	congruent, *or* is congruent to.	rt. ∡	right angles.	Cor.	corollary.
		△	triangle.	st.	straight.
>	is greater than.	▲	triangles.	rt.	right.
<	is less than.	rt. ▲	right triangles.	Def.	definition.
∴	hence, therefore, consequently.	∥	parallel.	alt.	alternate.
		∥s	parallels.	int.	interior.
⊥	perpendicular.	▱	parallelogram.	ext.	exterior.
⊥s	perpendiculars.	▱s	parallelograms.		

AXIOM, POSTULATE, AND THEOREM

30. An **axiom** is a statement admitted without proof to be true. It is a truth, received and assented to immediately.

31. AXIOMS.

1. Magnitudes that are equal to the same thing, or to equals, are equal to each other.

2. If equals are added to, or subtracted from, equals, the results are equal.

3. If equals are multiplied by, or divided by, equals, the results are equal.

[Doubles of equals are equal; halves of equals are equal.]

4. The whole is equal to the sum of all of its parts.

5. The whole is greater than any of its parts.

6. A magnitude may be displaced by its equal in any process. [Briefly called "substitution."]

7. If equals are added to, or subtracted from, unequals, the results are unequal in the same order.

8. If unequals are added to unequals in the same order, the results are unequal in that order.

9. If unequals are subtracted from equals, the results are unequal in the opposite order.

10. Doubles or halves of unequals are unequal in the same order. Also, **unequals multiplied by equals are unequal in the same order.**

11. If the first of three magnitudes is greater than the second, and the second is greater than the third, the first is greater than the third.

12. A straight line is the shortest line that can be drawn between two points.

13. Only one line can be drawn through a point parallel to a given line.

14. A geometrical figure may be moved from one position to another without any change in form or magnitude.

32. A **postulate** is something required to be done, the possibility of which is admitted without proof.

33. Postulates.

1. It is possible to draw a straight line from any point to any other point.

2. It is possible to extend (prolong or produce) a straight line indefinitely, or to terminate it at any point.

34. A geometric **proof** or **demonstration** is a logical course of reasoning by which a truth becomes evident.

35. A **theorem** is a statement that requires proof.

In the case of the preliminary theorems which follow, the proof is very simple; but as these theorems are not admitted without proof they cannot be classified with the axioms.

A **corollary** is a truth immediately evident, or readily established from some other truth or truths.

A **proposition**, in geometry, is the statement of a theorem to be proved or a problem to be solved.

Ex. 1. Draw an $\angle ABC$. In $\angle ABC$ draw line BD.
What does $\angle ABD + \angle DBC$ equal?
What does $\angle ABC - \angle ABD$ equal?

Ex. 2. In a rt. $\angle ABC$ draw line BD.
If $\angle ABD = 25°$, how many degrees are there in $\angle DBC$?

How many degrees are there in the complement of an angle of 38°? How many degrees are there in the supplement?

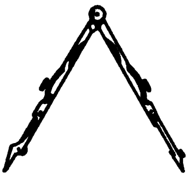
Ex. 3. Draw a straight line AB and take a point X on it.
What line does $AX + BX$ equal?
What line does $AB - BX$ equal?

Ex. 4. Draw a straight line AB and prolong it to X so that $BX = AB$. Prolong it so that $AB = AX$.

Historical Note. Probably as early as 3000 B.C. the Egyptians had some knowledge of geometric truths. The construction of the great pyramids required an acquaintance with the relations of geometry. This knowledge, however vague it may have been, was, according to Herodotus, employed in determining the amount of land washed away by the river Nile, during the reign of Rameses II (1400 B.C.).

The Greeks, however, were the first to study geometry as a logical science. They enunciated theorems and demonstrated them, they propounded problems and solved them as early as 300 B.C., and, in a crude way, two or three centuries earlier. To them belongs the credit of establishing a logical system of geometry that has survived, practically unchanged, for twenty centuries.

EXERCISES EMPLOYING THE TWO INSTRUMENTS OF GEOMETRY



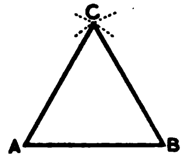
Aside from pencil and paper, the only instruments necessary for the construction of geometrical diagrams are the ruler and the compasses.

Ex. 1. It is required to draw an equilateral triangle upon a given line as base.

Suppose AB is the given base.

Required to draw an equilateral $\triangle$ upon it.

Using A as a center and AB as a radius, draw an arc. Using B as a center and AB as a radius, draw another arc cutting the first one at C. Draw AC and BC. The $\triangle ABC$ is an equilateral $\triangle$, and AB is its base.

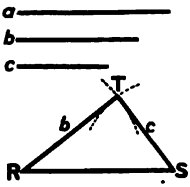


Ex. 2. It is required to draw a triangle having its three sides each equal to a given line.

Suppose the three given lines are a, b, c.

Required to draw a $\triangle$ having for its sides lines equal to a, b, c, respectively.

Draw a line $RS =$ to a. Using R as a center and b as a radius, draw an arc. Using S as a center and c as a radius, draw another arc cutting the first arc at T. Draw straight lines RT and ST. $\triangle RST$ is the $\triangle$ whose three sides are equal to the lines a, b, c, respectively.



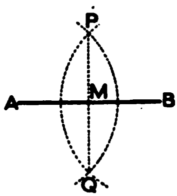
Ex. 3. It is required to find the midpoint of a given straight line.

Given the straight line AB.

Required to find its midpoint.

Using A and B as centers and a radius sufficiently long, draw two arcs, intersecting at P and Q.

Draw the straight line PQ cutting AB at M. Point M is the midpoint of AB.

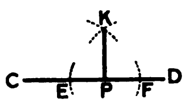


Ex. 4. It is required to draw a perpendicular to a line from a point within the line.

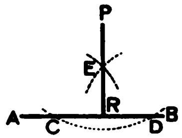
Given the line CD and point P in it.

Required to construct a $\perp$ to CD, at P.

Using P as center and any radius, draw two arcs cutting CD at E and F. Now using E and F as centers and a radius greater than before, draw two arcs intersecting at K. Draw KP. This line KP is $\perp$ to CD at P.



Ex. 5. It is required to draw a perpendicular to a line from a point without the line.

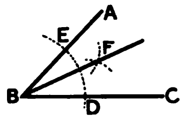


Given line AB and point P, without it.

Required to draw a $\perp$ to AB from P.

Using P as center and a sufficient radius, draw an arc cutting AB at C and D. Now using C and D as centers and a sufficient radius, draw two arcs intersecting at E. Draw PE, meeting AB at R. PR is the required $\perp$ to AB from P.

Ex. 6. It is required to bisect a given angle.



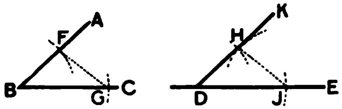
Given the $\angle ABC$.

Required to bisect it.

Using vertex B as a center and any radius, draw arc DE cutting BC at D and BA at E.

Using D and E as centers and a sufficient radius, draw arcs intersecting at F. Draw straight line BF. BF bisects the $\angle ABC$.

Ex. 7. It is required to construct, at a given point on a given line, an angle equal to a given angle.



Given line DE, point D in it, and $\angle B$.

Required to construct an $\angle$ at D, equal to $\angle B$.

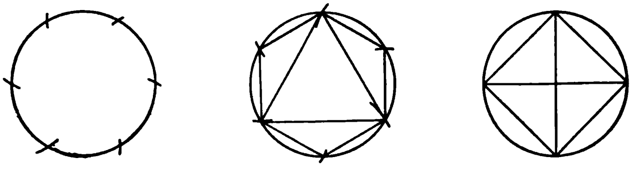
Using B as a center and with any two distances as radii, draw an arc cutting AB at F and another cutting BC at G.

Using D as a center and the same radii as before, draw one arc, and another arc cutting DE at J.

Draw the straight line FG. Using J as a center and FG as a radius, draw an arc cutting a former arc at H. Draw the straight lines HJ and DHK.

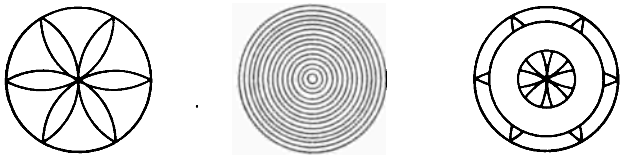
Now the $\angle KDE = \angle B$.

Ex. 8. By the use of ruler and compasses, draw the following figures:

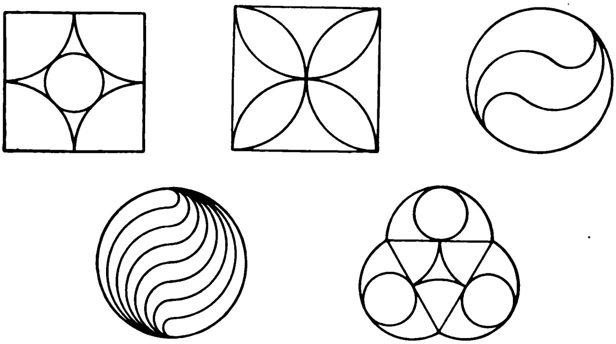


Ex. 9. Does it make any difference in these exercises, which lines are drawn first? In Ex. 7 and Ex. 8 explain the order of the lines drawn.

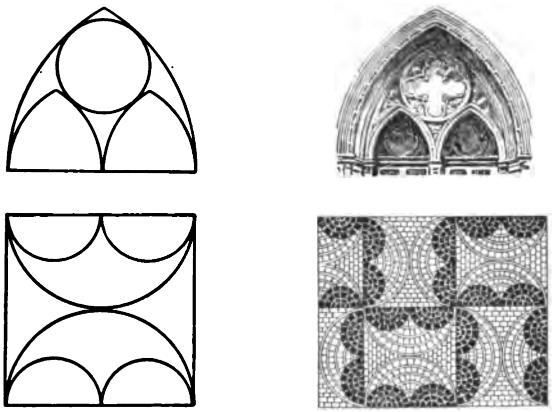
Ex. 10. Using the compasses only, draw the following figures:



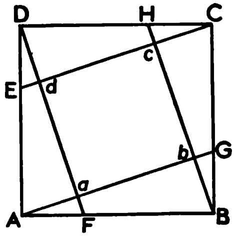
Ex. 11. Draw the following figures:



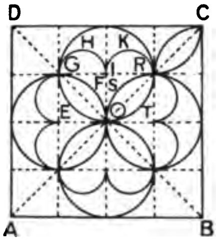
Ex. 12. Draw the first of each of these three pairs of figures. Can you explain the construction of the second figure in each pair?



In this figure, *ABCD* is a square. On the sides are measured the equal distances *AE* and *BF*, and *CG* and *DH*; then the lines *AG*, *BH*, *CE*, and *DF*, are drawn intersecting at *a*, *b*, *c*, *d*. The figure *abcd* is also a square. This figure is the basis of an Arabic design used for parquet floors, etc.

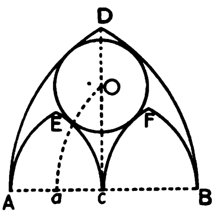


In this figure, which is the basis of a mosaic floor design, the radii of all complete circles equal one fourth of the side of the square *ABCD*. The radii of the semicircles *GHI*, *IKR*, etc., equal one eighth of the side of the square.



In this figure *ABD* is an equilateral arch, and *CD* is its altitude. The several centers used are *A*, of arc *BD* and arc *CE*; *B*, of arc *AD* and *CF*; *C*, of arcs *AE* and *BF*.

This figure is the basis of a common Gothic window design.



NOTE. The letters "Q.E.D." are often annexed at the end of a demonstration and stand for "*quod erat demonstrandum*," which means, "which was to be proved."

BOOK I

ANGLES, LINES, RECTILINEAR FIGURES

PRELIMINARY THEOREMS

36. A right angle is equal to half a straight angle.

Because of the definition of a right angle. (15.)

37. A straight angle is equal to two right angles. (36.)

38. Two straight lines can intersect in only one point.

Because they would coincide entirely if they had two common points. (4.)

39. Only one straight line can be drawn between two points. (4.)

40. A definite (limited or finite) **straight line can have only one midpoint.**

Because the halves of a line are equal.

41. All straight angles are equal.

Because they can be made to coincide. (26.)

42. All right angles are equal.

They are halves of straight angles. (36.)

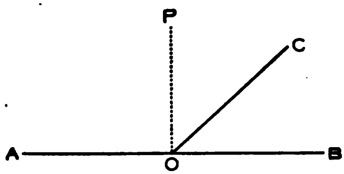
$\therefore$ they are equal. (Ax. 3.)

43. Only one perpendicular to a line can be drawn from a point in the line.

These right angles would not be equal if there were two perpendiculars. (42.)

44. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

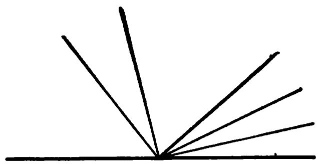
Because they together equal two rt. $\angle s$. (19.)



45. If two adjacent angles are supplementary, their exterior sides are in the same straight line.

Because their sum is two rt.$\angle s$ (19); or a straight $\angle$ (37). Hence the exterior sides are in the same straight line (17).

46. The sum of all the angles on one side of a straight line at a point equals two right angles. (Ax. 4 and 37.)



47. The sum of all the angles about a point in a plane is equal to four right angles. (46.)

48. Angles that have the same complement are equal. Or, complements of the same angle, or of equal angles, are equal.

Because equal angles subtracted from equal right angles leave equal angles. (Ax. 2.)

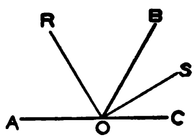
49. Angles that have the same supplement are equal. Or, supplements of the same angle, or of equal angles, are equal. (Ax. 2.)

50. If two angles are equal and supplementary, they are right angles.

Each is half a straight $\angle$; $\therefore$ each is a rt. $\angle$. (36.)

NOTE. A reference number usually indicates only the statement in **full face type** in the section referred to. In giving demonstrations the pupil should quote the correct reason for each statement.

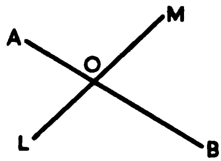
Ex. The bisectors of two supplementary adjacent angles are perpendicular to each other.



THEOREMS AND DEMONSTRATIONS

PROPOSITION I. THEOREM

51. If two straight lines intersect, the vertical angles are equal.



Given: Lines AB and LM intersecting at O, $\measuredangle s$ AOM and BOL, a pair of vertical $\measuredangle s$.

To Prove: $\angle AOM = \angle BOL$.

Proof: $\measuredangle s$ AOM and MOB are supplementary (44).

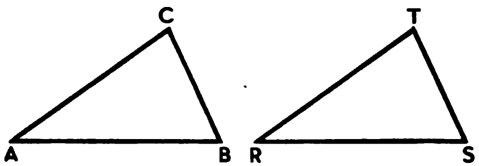
$\measuredangle s$ MOB and BOL are supplementary (44).

$\therefore \angle AOM = \angle BOL$. (49.)

Q.E.D.

PROPOSITION II. THEOREM

52. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.



Given: $\triangle s$ ABC and RST; $AB = RS$; $AC = RT$; $\angle A = \angle R$.

To Prove: $\triangle ABC$ is congruent to $\triangle RST$.

Proof: Place the $\triangle ABC$ upon the $\triangle RST$ so that $\angle A$ coincides with its equal $\angle R$.

AB falls upon RS and point B upon S ($AB = RS$).

AC falls upon RT and point C upon T ($AC = RT$).

$\therefore BC$ coincides with ST (39).

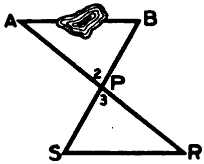
$\therefore$ the $\triangle s$ coincide and are congruent (26).

Q.E.D.

53. Corollary. Two right triangles are congruent if two legs of one are equal respectively to two legs of the other.

This is a corollary following immediately from 52.

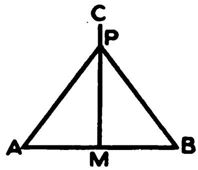
Ex. 1. If two triangles have two sides of one equal to two sides of the other, are the triangles necessarily congruent?



Ex. 2. Illustrate your answer to Ex. 1 by drawing two triangles.

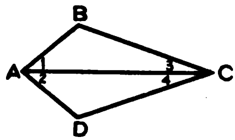
Ex. 3. Find the distance AB if there is an obstruction between A and B.

Method. Take a convenient point P, from which A and B are accessible. Measure AP and in same straight line mark point R such that $PR = AP$. Similarly find point S. Show that the two ▲ ABP and SRP are congruent, therefore the length of AB may be found by measuring RS.



Ex. 4. Prove that a point, P, in the perpendicular bisector MC of a line AB is equally distant from the ends of the line AB.

(Show that the ▲ AMP and BMP are congruent, (1) by using 52; and (2) by using 53.)



Ex. 5. If the line AC bisects $\angle BAD$, and $BA = AD$, prove that the triangles ABC and ADC are congruent, the line AC bisects $\angle BCD$, and BC equals CD.

Ex. 6. Prove that if a line from a vertex of a triangle perpendicular to the opposite side bisects that side, the triangle is isosceles.

Ex. 7. Draw two angles that are adjacent and not supplementary; adjacent and not complementary.

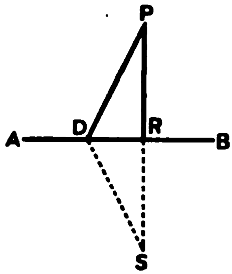
Ex. 8. Of two unequal angles which has the greater supplement?

Ex. 9. The complement of a certain angle added to the supplement of the same angle is 176°. Find the angle.

Ex. 10. What angle added to one fifth of its supplement equals a right angle?

Ex. 11. In the figure of 51, if $\angle AOM$ is 100°, how many degrees are there in each of the other angles at O?

PROPOSITION III. THEOREM



54. Only one perpendicular can be drawn to a line from an external point.

Given: $PR \perp$ to AB from P, and PD any other line from P to AB.

To Prove: PD is not $\perp$ to AB; that is, PR is the only $\perp$ from P to AB.

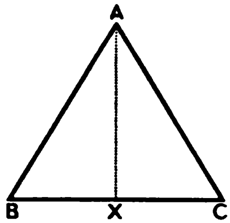
Proof: 1. Extend PR to S making $RS =$ to PR.

2. Draw DS.	
3. In rt. $\triangle$s PDR and SDR,	
$PR = RS$.	3. Construction.
4. $DR = DR$.	4. Identical.
5. $\therefore \triangle PDR$ is congruent to $\triangle SDR$.	5. Quote 53.
6. $\therefore \angle PDR = \angle SDR$.	6. Quote 27.
7. That is, $\angle PDR =$ half of $\angle PDS$.	
8. Now line PRS is a st. line.	8. Construction.
9. $\therefore$ line PDS is *not* a st. line.	9. Quote 39.
10. $\therefore \angle PDR$, half of $\angle PDS$, is *not* a rt. $\angle$.	10. Quote 36.
11. $\therefore PD$ is *not* $\perp$ to AB.	11. Quote 16.
That is, PR is the only $\perp$ from P to AB.	
Q.E.D.	

The preceding form of demonstration will serve to illustrate an excellent plan of writing the proofs. It will be observed that the *statements* appear at the left of the page and their *reasons* at the right. This arrangement will be found of great value in the saving of time, both for the pupil who writes the proofs and for the teacher who reads them.

Historical Note. The proof of the following theorem as given in the fifth proposition of Euclid's "The Elements," the most famous geometry that was ever written, was considered by the beginners as presenting great difficulties. The theorem was therefore called by the ancient teachers, the *pons asinorum*, or the bridge of the asses. Euclid discussed only magnitudes, not their numerical measures. Another note (p. 45) will tell more of the author of this renowned book.

Proposition IV. Theorem



55. The angles opposite the equal sides of an isosceles triangle are equal.

Given: $\triangle ABC$, $AB = AC$.

To Prove: $\angle B = \angle C$.

Proof: Suppose AX is drawn dividing $\angle BAC$ into two equal $\angle s$ and meeting BC at X. In the $\triangle s$ ABX and ACX,

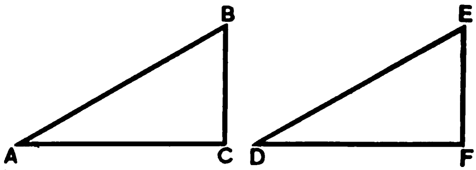
$AX = AX$	(Identical).
$AB = AC$	(Given).
$\angle BAX = \angle CAX$	(Const.).
$\therefore \triangle ABX$ is congruent to $\triangle ACX$	(52).
$\therefore \angle B = \angle C$	(27).

Q.E.D.

56. Corollary. An equilateral triangle is equiangular.

Proposition V. Theorem

57. Two right triangles are congruent if the hypotenuse and an adjoining angle of one are equal respectively to the hypotenuse and an adjoining angle of the other.



Given: Rt. $\triangle s$ ABC and DEF; $AB = DE$; and $\angle A = \angle D$.

To Prove: $\triangle ABC$ is congruent to $\triangle DEF$.

Proof: Place $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and AC falls along DF.

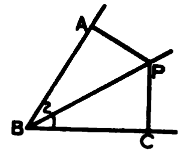
Then AB coincides with DE and point B falls exactly on E $(AB = DE)$.

Now, from point E, BC and EF are both $\perp$ to DF (16).

$\therefore$ BC coincides with EF (54).

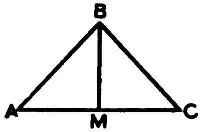
$\therefore$ $\triangle ABC$ is congruent to $\triangle DEF$ (26).

Q.E.D.



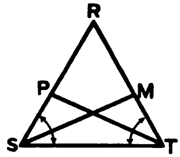
Ex. 1. In the adjoining diagram, if $\angle 1 = \angle 2$, prove the right triangles congruent.

Ex. 2. By use of 27 prove $PA = PC$.



Ex. 3. Prove that the line, BM, from the vertex of an isosceles triangle, ABC, and perpendicular to the base, bisects the vertex angle and also bisects the base.

(First prove the two right $\triangle$ congruent, and then use 27.)



Ex. 4. Prove that the perpendiculars, SM and TP, upon the equal sides of an isosceles triangle, RST, from the opposite vertices, S and T, are equal.

Two ways: (*a*) Show rt. $\triangle$ PST and MST are congruent; or (*b*) show rt. $\triangle$ RSM and RPT are congruent.

Ex. 5. Prove that the bisector of the vertex angle of an isosceles triangle bisects the base at right angles.

58. Homologous parts. Triangles are proved congruent in order that their homologous sides or homologous angles may be proved equal.

59. Auxiliary lines. Often it is impossible to give a simple demonstration without drawing lines not described in the hypothesis. Such lines, used only for the proof, are usually dotted in order to distinguish them from the lines mentioned in the hypothesis and the conclusion.

60. Elements of a theorem. Every theorem contains two parts. The one is assumed to be true; the other results from this assumption. The one part contains the given conditions; the other part states the resulting truth.

The assumed part of a theorem is called the **hypothesis.**

The part of the theorem which is to be proved true is the **conclusion.**

Often the hypothesis is a clause introduced by the word "if." When this conjunction is omitted, the subject of the sentence is known and its qualities, described in the qualifying words, constitute the "given conditions." Thus, in the theorem of 52, the assumed part follows the word "if," and the truth to be proved is: "Two triangles are equal."

The **converse of a theorem** is the theorem obtained by interchanging the hypothesis and the conclusion of the original theorem. Consult 44 and 45, 55 and 114.

NOTE. Every theorem having a simple hypothesis and a simple conclusion has a converse, but only a few of these converse theorems are true.

61. Elements of a demonstration. All correct demonstrations should consist of certain distinct parts, namely:

1. Full statement of the **given conditions** as applied to a particular figure.

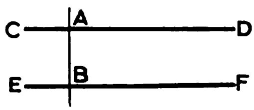
2. Full statement of the truth which it is required **to prove.**

3. The **proof** — a series of successive statements, for each of which a valid reason should be quoted. (The drawing of auxiliary lines is sometimes essential.)

4. The **conclusion** declared to be true.

PROPOSITION VI. THEOREM

62. Two lines in the same plane and perpendicular to the same line are parallel.



Given: *CD* and *EF* in same plane and both ⊥ to *AB*.

To Prove: *CD* and *EF* ∥.

Proof. If *CD* and *EF* were not ∥, they would meet if sufficiently prolonged.

CD and EF are both $\perp$ to AB (Given).

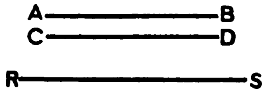
$\therefore$ there would be two lines $\perp$ to AB from the point of meeting. But this is impossible (54).

$\therefore$ CD and EF do not meet and are $\parallel$ (21).

Q.E.D.

Proposition VII. Theorem

63. Two lines in the same plane and parallel to the same line are parallel.



Given: $AB \parallel$ to RS, and $CD \parallel$ to RS, in the same plane.

To Prove: $AB \parallel$ to CD.

Proof: If AB and CD were not $\parallel$, they would meet if sufficiently prolonged.

AB and CD are both $\parallel$ to RS (Given).

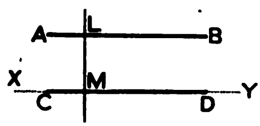
$\therefore$ there would be two lines $\parallel$ to RS through the point of meeting. But this is impossible (Ax. 13).

$\therefore$ AB and CD do not meet and are $\parallel$ (21).

Q.E.D.

Proposition VIII. Theorem

64. If a line is perpendicular to one of two parallels, it is perpendicular to the other also.



Given: $LM \perp$ to AB and $AB \parallel$ to CD.

To Prove: $LM \perp$ to CD.

Proof: Suppose XY is drawn through $M \perp$ to LM

XY is $\parallel$ to AB (62).

But CD is $\parallel$ to AB (Given).

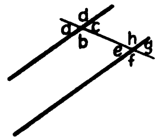
And CD and XY both contain point M (Const.).

$\therefore$ CD and XY coincide (Ax. 13).

But LM is $\perp$ to XY (Const.).

That is, LM is $\perp$ to CD Q.E.D.

65. If one line cuts other lines, it is called a **transversal**. Angles are formed at the several intersections, as follows :



b, *c*, *e*, *h* are **interior angles.**

a, *d*, *f*, *g* are **exterior angles.**

b and *h*, *c* and *e*, *a* and *g*, *d* and *f* (on opposite sides of the transversal) are **alternate angles.**

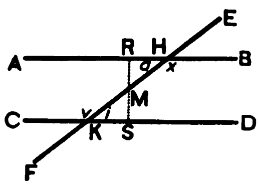
b and *h*, *c* and *e* are **alternate interior angles.**

a and *g*, *d* and *f* are **alternate exterior angles.**

a and *e*, *d* and *h*, *b* and *f*, *c* and *g* are **corresponding angles.**

Proposition IX. Theorem

66. If a transversal intersects two parallels, the alternate interior angles are equal.



Given: $AB \parallel$ to CD; transversal EF cutting the $\parallel$s at H and K.

To Prove: $\angle a = \angle i$ and $\angle x = \angle v$.

Proof: Suppose through M, the midpoint of HK, RS is drawn $\perp$ to AB. Then RS is $\perp$ to CD. (64).

In rt. $\triangle$s RMH and KMS, $HM = KM$ (Const.).

$\angle RMH = \angle KMS$ (51).

$\therefore \triangle RMH$ is congruent to $\triangle KMS$ (57).

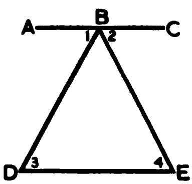
$\therefore \angle a = \angle i$ (27).

Again $\angle x$ is the supplement of $\angle a$ (44).

Also $\angle v$ is the supplement of $\angle i$ (44).

$\therefore \angle x = \angle v$ (49).

Q.E.D.

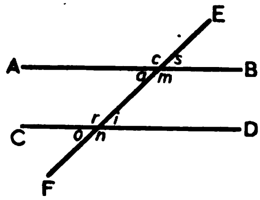


Ex. 1. If a line through the vertex of an isosceles triangle is parallel to the base, it makes equal angles with the sides of the triangle.

Ex. 2. If from each point at which a transversal intersects two parallels a perpendicular to the other parallel is drawn, two congruent right triangles are formed.

PROPOSITION X. THEOREM

67. If a transversal intersects two parallels, the corresponding angles are equal.



Given: $AB \parallel$ to CD; transversal EF cutting the ∥s and forming the 8 ∠s.

To Prove: $\angle s = \angle i$; $\angle c = \angle r$; $\angle o = \angle a$; $\angle n = \angle m$.

Proof: I. $\angle s = \angle a$ (51).

$\angle a = \angle i$ (66).

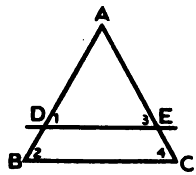
$\therefore \angle s = \angle i$ (Ax. 1).

II. $\angle c = \angle m$ (?).

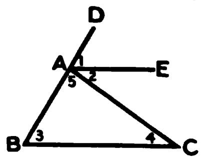
$\angle m = \angle r$ (?).

$\therefore \angle c = \angle r$ (?).

Etc. Q.E.D.



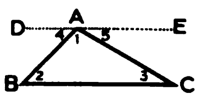
Ex. 1. If a line intersects the equal sides of an isosceles triangle and is parallel to the third side, the triangle formed has two equal angles.



Ex. 2. In the adjoining figure, if AE is parallel to BC, prove that $\angle 1 + \angle 2$ or $\angle CAD = \angle 3 + \angle 4$.

Ex. 3. In the same figure prove that

$$\angle 3 + \angle 4 + \angle 5 = 2 \text{ rt. } \angle s.$$



Ex. 4. By drawing a line, DE, through the vertex A, of a triangle ABC, parallel to BC, prove that the sum of the angles of any triangle is equal to two right angles.

Ex. 5. If the equal sides of an isosceles triangle are prolonged through the vertex, and a line is drawn parallel to the base, cutting these prolongations, this triangle formed has two equal angles.

Ex. 6. If two sides of any triangle are prolonged beyond the third side, and a line, parallel to the third is drawn cutting these prolongations, two mutually equiangular triangles are formed.

PROPOSITION XI. THEOREM

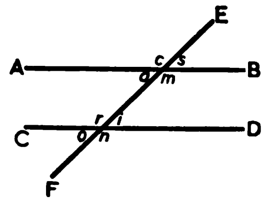
68. If a transversal intersects two parallels, the alternate exterior angles are equal.

Given: (?) **To Prove:** (?)

Proof: $\angle c = \angle r$ (?); $\angle r = \angle n$ (?). $\therefore \angle c = \angle n$ (?) etc.

Q.E.D.

PROPOSITION XII. THEOREM



69. If a transversal intersects two parallels, the interior angles on the same side of the transversal are supplementary.

Given: (?)

To Prove: $\angle a + \angle r = 2$ rt. $\angle$s. etc.

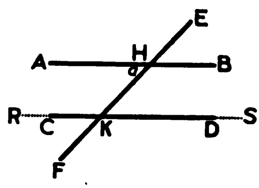
Proof: $\angle a + \angle m = 2$ rt $\angle$s (46).

But $\angle m = \angle r$ (66).

$\therefore \angle a + \angle r = 2$ rt. $\angle$s etc. (Ax. 6). Q.E.D.

PROPOSITION XIII. THEOREM

70. If a transversal intersects two lines and the alternate interior angles are equal, the lines are parallel. [Converse of 66.]



Given: *AB* and *CD* two lines; transversal *EF* cutting them at *H* and *K*, respectively; $\angle a = \angle HKD$.

To Prove: *CD* ∥ to *AB*.

Proof: Through *K* suppose *RS* is drawn ∥ to *AB*.

Then $\angle a = \angle HKS$ (66).

But $\angle a = \angle HKD$ (Hyp.).

$\therefore \angle HKS = \angle HKD$ (Ax. 1).

That is, *KD* and *KS* coincide, and *CD* and *RS* are the same line. $\therefore$ *CD* is ∥ to *AB*.

(Because it coincides with *RS*, which is ∥ to *AB*.) Q.E.D.

PROPOSITION XIV. THEOREM

71. If a transversal intersects two lines and the corresponding angles are equal, the lines are parallel. [Converse of 67.]

Given: AB and CD cut by EF; $\angle c = \angle r$. (Fig. in 69.)

To Prove: $AB \parallel$ to CD.

Proof:

$\angle c = \angle m$	(51).
$\angle c = \angle r$	(Hyp.).
$\therefore \angle m = \angle r$	(Ax. 1).
$\therefore AB$ is $\parallel$ to CD.	(70).

Q.E.D.

PROPOSITION XV. THEOREM

72. If a transversal intersects two lines and the alternate exterior angles are equal, the lines are parallel. [Converse of 68.]

Given: AB and CD cut by EF, and $\angle c = \angle n$.

To Prove: $AB \parallel$ to CD.

Proof:

$\angle c = \angle m$	(51).
$\angle c = \angle n$	(Hyp.).
$\therefore \angle m = \angle n$	(Ax. 1).
$\therefore AB$ is $\parallel$ to CD	(71). Q.E.D.

PROPOSITION XVI. THEOREM

73. If a transversal intersects two lines and the interior angles on the same side of the transversal are supplementary, the lines are parallel. [Converse of 69.]

Given: AB and CD cut by EF and $\angle a + \angle r = 2$ rt. $\angle$s.

To Prove: $AB \parallel$ to CD.

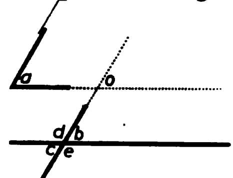
Proof:

$\angle a$ is the supplement of $\angle c$	(44).
$\angle a$ is the supplement of $\angle r$	(Hyp.).
$\therefore \angle c = \angle r$	(49).
$\therefore AB$ is $\parallel$ to CD	(71). Q.E.D.

PROPOSITION XVII. THEOREM

74. If two angles have their sides parallel each to each, the angles are equal or supplementary.

NOTE. There are three cases: (I) the pairs of sides extending in the same two directions from the vertices; (II) the pairs of sides extending in opposite directions from the vertices; (III) one pair extending in the same direction and the other pair in opposite directions from the vertices.



I. **Given:** $\angle a$ and $\angle b$, with their sides $\parallel$ each to each and extending in the *same directions* from their vertices.

To Prove: $\angle a = \angle b$.

Proof: If the non-parallel lines do not meet, produce them to meet, forming $\angle o$.

$\angle a = \angle o$ (67).

$\angle o = \angle b$ (67).

$\therefore \angle a = \angle b$ (Ax. 1).

Q.E.D.

II. **Given:** $\angle a$ and $\angle c$ with their sides $\parallel$ each to each and extending in *opposite directions* from their vertices.

To Prove: $\angle a = \angle c$

Proof:

$\angle a = \angle b$ (Proved in I).

$\angle b = \angle c$ (51).

$\therefore \angle a = \angle c$ (Ax. 1).

Q.E.D.

III. **Given:** $\angle a$ and $\angle d$ with their sides $\parallel$; one pair extending in the *same direction* and the other pair in *opposite directions* from their vertices.

To Prove: $\angle a$ is supplementary to $\angle d$.

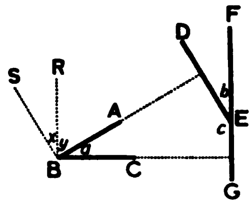
Proof: $\angle b$ is supplementary to $\angle d$ (44).

But $\angle a = \angle b$ (Proved in I).

Substituting, $\angle a$ is supp. to $\angle d$. (Ax. 6). Q.E.D.

The proof that $\angle a$ and $\angle e$ are supplementary is the same.

PROPOSITION XVIII. THEOREM



75. If two angles have their sides perpendicular each to each, the angles are equal or supplementary.

I. **Given:** $\measuredangle$ *a* and *b* with sides $\perp$ each to each.

To Prove: $\angle a = \angle b$.

Proof: At *B* suppose *BR* is drawn $\perp$ to *BC* and *BS* $\perp$ to *AB*.

BR is $\parallel$ to *FE* and *BS* is $\parallel$ to *DE* (64).

$\therefore \angle x = \angle b$ (74).

Now $\angle a$ is the complement of $\angle y$ (19).

Also $\angle x$ is the complement of $\angle y$ (19).

$\therefore \angle a = \angle x$ (48).

$\therefore \angle a = \angle b$ (Ax. 1).

II. **Given:** $\measuredangle$ *a* and *c* with sides $\perp$ each to each.

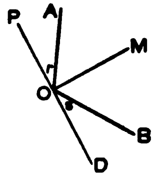
To Prove: $\angle a$ and $\angle c$ supplementary.

Proof: $\angle b$ and $\angle c$ are supplementary (44).

But $\angle a = \angle b$ (Proved in I).

Substituting, $\angle a$ and $\angle c$ are supplementary (Ax. 6).

Q.E.D.



Ex. 1. If a line is drawn through the vertex of an angle and perpendicular to the bisector of the angle, it makes equal angles with the sides.

Ex. 2. Draw figure for Proposition XVII showing $\angle b$ within $\angle a$, and prove the theorem.

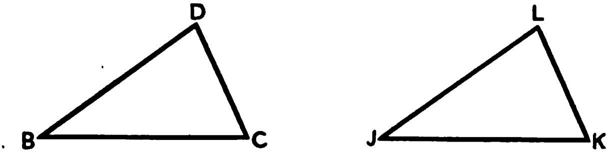
Ex. 3. Draw figure for Proposition XVIII showing $\angle b$ without $\angle a$, and prove the theorem.

Ex. 4. Prove Proposition XVIII if the given angles have the same vertex; if the vertex of one angle is on a side of the other.

Ex. 5. In figure of 75, if $\angle a = 40°$, tell how many degrees there are in $\measuredangle$ *b*, *c*, *x*, *y*, and *SBC*.

Proposition XIX. Theorem

76. Two triangles are congruent if a side and the two angles adjoining it in the one are equal respectively to a side and the two angles adjoining it in the other.



Given: $\triangle BCD$ and JKL; $BC = JK$; $\angle B = \angle J$; $\angle C = \angle K$.

To Prove: $\triangle BCD$ is congruent to $\triangle JKL$.

Proof: Place $\triangle BCD$ upon $\triangle JKL$ so that BC coincides with its equal, JK.

BD falls on JL (Because $\angle B$ is given $=$ to $\angle J$).

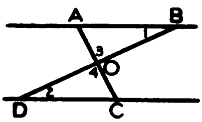
CD falls on KL (Because $\angle C$ is given $=$ to $\angle K$).

Then point D, which falls on both the lines JL and KL, falls at their intersection, L (38).

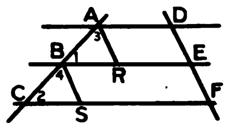
$\therefore$ the $\triangle$ are congruent (26).

Q.E.D.

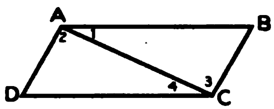
77. Corollary. **Two right triangles are congruent if a leg and the adjoining acute angle of one are equal respectively to a leg and the adjoining acute angle of the other.**



Ex. 1. If AB is $\parallel$ to DC and point O bisects transversal BD, prove that it also bisects transversal AC.



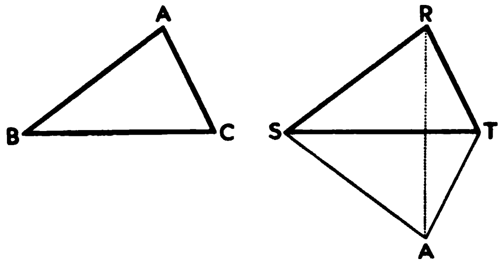
Ex. 2. In the accompanying figure, the three $\parallel$s are cut by two transversals, $AB = BC$, AR and BS are $\parallel$ to DF. Prove that $AR = BS$.



Ex. 3. In the accompanying figure, AB is $\parallel$ to DC and AD is $\parallel$ to BC. Prove that the $\triangle$ are congruent.

PROPOSITION XX. THEOREM

78. Two triangles are congruent, if the three sides of one are equal respectively to the three sides of the other.



Given: $\triangle s$ ABC and RST; $AB = RS$; $AC = RT$; $BC = ST$.

To Prove: $\triangle RST$ is congruent to $\triangle ABC$.

Proof: Place $\triangle ABC$ in the position of $\triangle AST$, so that the longest equal sides, BC and ST coincide, and A is opposite ST from R. Draw RA.

Now	$RS = AS$	(Hyp.).
	$\therefore \triangle ASR$ is an isosceles $\triangle$	(Def.).
Also	$TR = TA$	(Hyp.).
	$\therefore \triangle ATR$ is an isosceles $\triangle$	(Def.).
	$\therefore \angle SRA = \angle SAR$	(55).
Also	$\angle TRA = \angle TAR$	(55).
Adding,	$\angle SRT = \angle SAT$	(Ax. 2).
	$\therefore \triangle RST$ is congruent to $\triangle AST$	(52).

That is, by substituting, $\triangle RST$ is congruent to $\triangle ABC$ (Ax. 6).

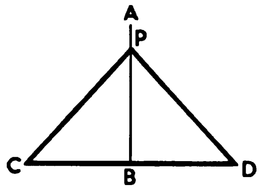
Q.E.D

Ex. In the figure of Ex. 3 on the opposite page, if the opposite sides are equal, prove them parallel.

79. The **distance** from one point to another is the length of the straight line joining the two points.

Proposition XXI. Theorem

80. Any point in the perpendicular bisector of a line is equally distant from the extremities of the line.



Given: $AB \perp$ to CD at its midpoint, B; P any point in AB; PC and PD.

To Prove: $PC = PD$.

Proof: In the rt. $\triangle$s PBC and PBD,

$PB = PB$ (Identical).

$BC = BD$ (Hyp.).

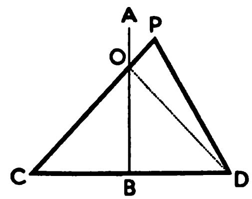
$\therefore \triangle PBC$ is congruent to $\triangle PBD$ (53).

$\therefore PC = PD$. (27).

Q.E.D.

Proposition XXII. Theorem

81. Any point not in the perpendicular bisector of a line is not equally distant from the extremities of the line.



Given: $AB \perp$ bisector of CD; P any point not in AB; PC and PD.

To Prove: $PC \neq PD$.

Proof: Either PC or PD will cut AB. Suppose PC cuts AB at O. Draw OD.

$DO + OP > PD$ (Ax. 12).

But $CO = DO$ (80).

Substituting, $CO + OP > PD$ (Ax. 6).

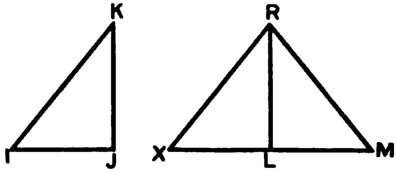
That is, $PC > PD$ or $PC \neq PD$. Q.E.D.

82. Corollary. **If a point is equally distant from the extremities of a line, it is in the perpendicular bisector of the line.** (80 and 81.)

83. Corollary. **Two points each equally distant from the extremities of a line determine the perpendicular bisector of the line.** (82 and 4.)

PROPOSITION XXIII. THEOREM

84. Two right triangles are equal if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.



Given: rt. $\triangle$ IJK and LMR; $KI = RM$; $KJ = RL$.

To Prove: $\triangle IJK = \triangle LMR$.

Proof: Place $\triangle IJK$ in the position of $\triangle XLR$ so that the equal sides, KJ and RL, coincide, and I is at X, opposite RL from M.

Now	$\angle$s RLM and RLX are supplementary	(19).
	$\therefore XLM$ is a straight line	(45).
	$\therefore$ figure XMR is a $\triangle$	(23).
Now	$RX = RM$	(Hyp.).
	$\therefore \triangle XMR$ is isosceles	(Def.).
	$\therefore \angle X = \angle M$	(55).
	$\therefore \triangle XLR$ is congruent to $\triangle LMR$	(57).
That is,	$\triangle IJK$ is congruent to $\triangle LMR$	(Ax. 6).

Q.E.D.

85. COROLLARY. **The perpendicular from the vertex of an isosceles triangle to the base bisects the base, and bisects the vertex angle.**

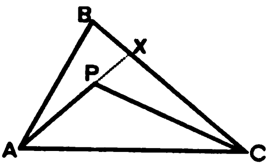
In the congruent right $\triangle$ of 84,	$XL = LM$	(27).
Also	$\angle XRL = MRL$	(27).

Q.E.D.

Ex. In the figure of 84, if $XL = LM$, prove by two methods that $XR = RM$.

PROPOSITION XXIV. THEOREM

86. The sum of two sides of a triangle is greater than the sum of two lines drawn to the extremities of the third side, from any point within the triangle.



Given: P, any point in $\triangle\ ABC$; lines PA and PC.

To Prove: $AB + BC > AP + PC$.

Proof: Extend AP to meet BC at X.

$$AB + BX > AP + PX \qquad \text{(Ax. 12).}$$
$$CX + PX > PC \qquad \text{(Ax. 12).}$$

Adding, $AB + \overline{BX + CX} + PX > AP + PC + PX$ (Ax. 8).

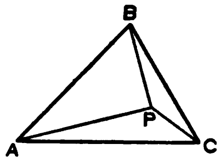
Subtract $PX = PX$

$\therefore AB + BC > AP + PC$ (Ax. 7). Q.E.D.

Ex. If from any point within a triangle lines are drawn to the three vertices:

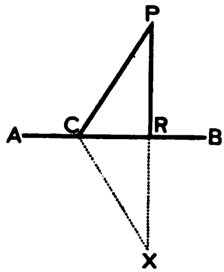
(1) their sum is less than the sum of the sides of the triangle.

(2) their sum is greater than half the sum of the sides of the triangle.



PROPOSITION XXV. THEOREM

87. The perpendicular is the shortest line that can be drawn from a point to a straight line.



Given: $PR \perp$ to AB; PC not $\perp$.

To Prove: $PR < PC$.

Proof: Extend PR to X, making $RX =$ to PR. Draw CX.

(1) $PR + RX < PC + CX$ (Ax. 12).

But AR is $\perp$ bisector of PX (Const.).

$\therefore CX = PC$ (80).

Also $RX = PR$ (Const.).

$\therefore$ Substituting in (1), $PR + PR < PC + PC$ (Ax. 6).

That is, $2\,PR < 2\,PC$.

$\therefore PR < PC$ (Ax. 10). Q.E.D.

PROPOSITION XXVI. THEOREM

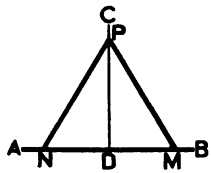
88. If from any point in a perpendicular to a line two oblique lines are drawn,

I. Oblique lines cutting off equal distances from the foot of the perpendicular are equal.

II. Equal oblique lines cut off equal distances. [Converse.]

III. Oblique lines cutting off unequal distances are unequal, and that one which cuts off the greater distance is the greater.

IV. Unequal oblique lines cut off unequal distances from the foot of the perpendicular, and the longer oblique line cuts off the greater distance. [Converse.]



I. **Given:** $CD \perp$ to AB; $ND = MD$; oblique lines PN and PM.

To Prove: $PN = PM$.

Proof: In the rt. $\triangle$s PDN and PDM, $PD = PD$ (Iden.).

Also $ND = DM$ (Hyp.).

$\therefore \triangle PDN$ is congruent to $\triangle PDM$ (53).

$\therefore PN = PM$ (27).

Q.E.D.

II. **Given:** $CD \perp$ to AB; $PN = PM$.

To Prove: $DN = DM$.

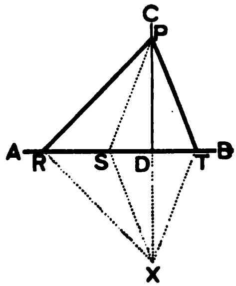
Proof: In the rt. $\triangle$s PDN and PDM, $PD = PD$ (Iden.).

Also $PN = PM$ (Hyp.).

$\therefore \triangle PDN$ is congruent to $\triangle PDM$ (84).

$\therefore DN = DM$ (27).

Q.E.D.



III. **Given:** $CD \perp$ to AB; oblique lines PR and PT:

Also $RD > DT$.

To Prove: $PR > PT$.

Proof: Because $DR > DT$, we may take DS (on DR) = to DT. Draw PS. Extend PD to X, making $DX =$ to PD. Draw RX and SX. Now AD is the $\perp$ bisector of PX and CD is the $\perp$ bisector of ST (Const.).

$PR + RX > PS + SX$ (86).

But $RX = PR$, and $SX = PS = PT$ (80).

Substituting, $PR + PR > PS + PS$ (Ax. 6).

That is, $2PR > 2PS$

$\therefore PR > PS$ (Ax. 10).

Substituting, $PR > PT$ (Ax. 6). Q.E.D.

IV. **Given:** [Use no dotted lines.] $CD \perp$ to AB; oblique lines PR and PT; $PR > PT$.

To Prove: $DR > DT$.

Proof: It is evident that $DR < DT$, or $DR = DT$, or $DR > DT$.

1st: If $DR < DT$, $PR < PT$ (By III).

But $PR > PT$ (Hyp.).

$\therefore DR$ is *not* $< DT$

2d: If $DR = DT$, $PR = PT$ (By I).

But $PR > PT$ (Hyp.).

$\therefore DR$ is *not* $= DT$

3d: Therefore, the only possibility is that $DR > DT$.

Q.E.D.

89. Corollary. From an external point it is not possible to draw three equal straight lines to a given straight line.

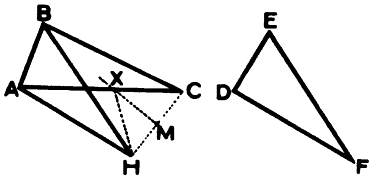
90. The method of demonstration employed in 88, IV, is called the **method of exclusion.**

It consists in making all possible suppositions, leaving the probable one last, and then proving all these suppositions impossible, except the last, which must necessarily be true.

The method of proving the individual steps is called *reductio ad absurdum* (reduction to an absurd or impossible conclusion). This method consists in assuming as false the truth to be proved and then showing that this assumption leads to a conclusion altogether contrary to known truth or the given hypothesis. (Examine the last preceding proof.) This is sometimes called the **indirect method.** The theorems of 62 and 63 are demonstrated by a single use of this method.

Proposition XXVII. Theorem

91. If two triangles have two sides of one equal to two sides of the other, but the included angle in the first greater than the included angle in the second, the third side of the first is greater than the third side of the second.



Given: $\triangle$ ABC, DEF;
$AB = DE$; $BC = EF$;
$\angle ABC > \angle E$.

To Prove: $AC > DF$.

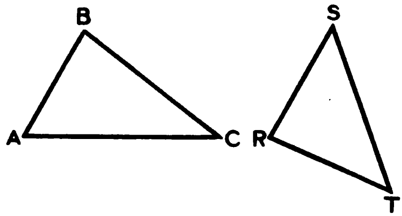
Proof: Place the $\triangle DEF$ upon $\triangle ABC$ so that side DE coincides with its equal AB, $\triangle DEF$ taking the position of $\triangle ABH$. There remains an $\angle HBC$. (Hyp.).

Draw HC and suppose its $\perp$ bisector, MX, to be erected, meeting AC at X. Draw HX.

Now $\qquad HX = XC \qquad$ (80).
Also $\qquad AX + XH > AH \qquad$ (Ax. 12).
Substituting, $\qquad AX + XC$, or $AC > AH \qquad$ (Ax. 6).
Substituting, $\qquad AC > DF \qquad$ (Ax. 6). Q.E.D.

Proposition XXVIII. Theorem

92. If two triangles have two sides of one equal to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second. [Converse.]



Given: $\triangle ABC$ and RST; $AB = RS$; $BC = ST$; $AC > RT$.

To Prove: $\angle B > \angle S$.

Proof: $\angle B < \angle S$, or $\angle B = \angle S$, or $\angle B > \angle S$.

1. If $\angle B < \angle S$, $AC < RT$ (91).
But $AC > RT$ (Hyp.).
$\therefore \angle B$ is *not* $< \angle S$.

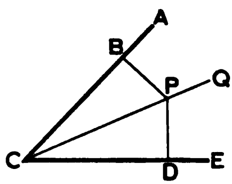
2. If $\angle B = \angle S$, the $\triangle$ are congruent (52).
$\therefore AC = RT$ (27).
But $AC > RT$ (Hyp.).
$\therefore \angle B \neq \angle S$.

3. $\therefore$ the only possibility is that $\angle B > \angle S$. Q.E.D.

93. The **distance** from a point to a line is the length of the perpendicular from the point to the line. If the perpendiculars from a point to two lines are equal, the point is said to be **equally distant** from the lines.

Proposition XXIX. Theorem

94. Every point in the bisector of an angle is equally distant from the sides of the angle.



Given: $\angle ACE$; bisector CQ; point P in CQ; distances PB and PD.

To Prove: $PB = PD$.

Proof: $\triangle$ *PBC* and *PDC* are rt. $\triangle$.

In rt. $\triangle$ *PBC* and *PDC*, $PC = PC$ (Iden.).

$\angle PCB = \angle PCD$ (Hyp.).

$\therefore \triangle PBC$ is congruent to $\triangle PDC$ (57).

$PB = PD$ (27). Q.E.D.

Proposition XXX. Theorem

95. Every point equally distant from the sides of an angle is in the bisector of the angle.

Given: $\angle ACE$; *P*, a point, such that $PB = PD$ (distances); *CQ*, a line from vertex of the angle, and containing *P*.

To Prove: $\angle ACQ = \angle ECQ$.

Proof: $\triangle$ *PBC* and *PDC* are rt. $\triangle$ (93).

In rt. $\triangle$ *PBC* and *PDC*, $PC = PC$ (Iden.).

$PB = PD$ (Hyp.)

$\therefore \triangle PBC$ is congruent to $\triangle PDC$ (84).

$\therefore \angle ACQ = \angle ECQ$ (27). Q.E.D.

96. Corollary. **If a point is not equally distant from the sides of an angle, it is not in the bisector of the angle.**

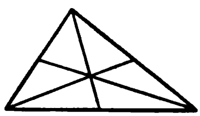
(If it were in the bisector, it would be equally distant.)

97. Corollary. **The vertex of an angle and a point equally distant from its sides determine the bisector of the angle.**

98. The **altitude** of a triangle is the perpendicular from any vertex to the opposite side (prolonged if necessary). A triangle has three altitudes.

The **bisector** of an angle of a triangle is the line dividing any angle into two equal angles. A triangle has three bisectors of its angles.

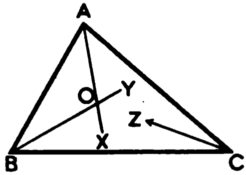
The **median** of a triangle is the line drawn from any vertex to the midpoint of the opposite side. A triangle has three medians.



THE THREE MEDIANS

Proposition XXXI. Theorem

99. The bisectors of the angles of a triangle meet in a point which is equally distant from the sides.



Given: $\triangle ABC$, AX bisecting $\angle A$, BY and CZ the other bisectors.

To Prove: AX, BY, CZ meet in a point equally distant from AB, AC, and BC.

Proof: Suppose that AX and BY intersect at O.

O, in AX, is equally distant from AB and AC (94).

O, in BY, is equally distant from AB and BC (?).

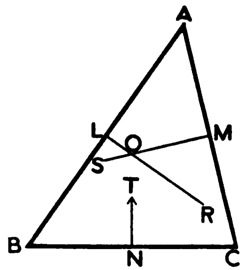
$\therefore$ point O is equally distant from AC and BC (Ax. 1).

$\therefore O$ is in bisector CZ (95).

That is, all three bisectors meet at O, and O is equally distant from the three sides. Q.E.D.

Proposition XXXII. Theorem

100. The three perpendicular bisectors of the sides of a triangle meet in a point which is equally distant from the vertices.



Given: $\triangle ABC$; LR, MS, NT, the three $\perp$ bisectors.

To Prove: LR, MS, NT meet in a point equally distant from A, B, C.

Proof: Suppose that LR and MS intersect at O.

O, in LR, is equally distant from A and B (80).

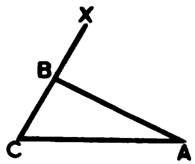
O, in MS, is equally distant from A and C (?).

$\therefore$ point O is equally distant from B and C (Ax. 1).

$\therefore O$ is in $\perp$ bisector NT (82).

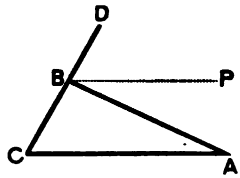
That is, all three $\perp$ bisectors meet at O, and O is equally distant from A and B and C. Q.E.D.

101. An **exterior angle** of a triangle is an angle formed outside the triangle, between one side of the triangle and another side prolonged. [$\angle ABX$.]



The angles within the triangle at the other vertices are the **opposite interior** angles. [$\angle A$ and $\angle C$.]

PROPOSITION XXXIII. THEOREM



102. An exterior angle of a triangle is equal to the sum of the opposite interior angles.

Given: $\triangle ABC$; exterior $\angle ABD$.

To Prove: $\angle ABD = \angle A + \angle C$.

Proof: Suppose BP to be drawn through $B \parallel$ to AC.

$\angle ABD = \angle ABP + \angle PBD$ (Ax. 4).

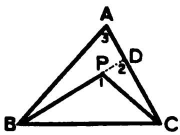
But $\angle ABP = \angle A$ (66).

Also $\angle PBD = \angle C$ (67).

$\therefore \angle ABD = \angle A + \angle C$ (Ax. 6). Q.E.D.

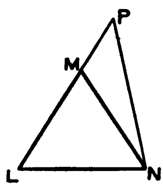
103. COROLLARY. **An exterior angle of a triangle is greater than either of the opposite interior angles.** (Ax. 5.)

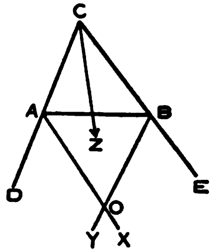
Ex. 1. If lines are drawn from any point within a triangle to two vertices of the triangle, they include an angle greater than the third angle of the triangle.



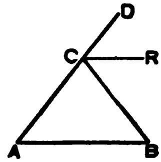
[Notice that $\angle 1$ is an ext. $\angle$ of $\triangle CDP$ and $\angle 2$ is an ext. $\angle$ of $\triangle ABD$.]

Ex. 2. If the side LM, of equilateral triangle LMN, is produced to P, and PN is drawn, $\angle P < \angle L$.





Ex. 3. The bisectors of two exterior angles of a triangle and of the interior angle at the third vertex meet in a point.

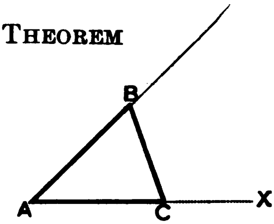


Ex. 4. The line through the vertex of an isosceles triangle, parallel to the base, bisects the exterior angle.

Ex. 5. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

Proof: $\angle DCB = 2\angle A$ (?) $= 2\angle DCR$ (?).

Proposition XXXIV. Theorem



104. The sum of the angles of any triangle is two right angles; that is, 180°.

Given: $\triangle ABC$.

To Prove:

$$\angle A + \angle B + \angle ACB = 2 \text{ rt. } \measuredangle s = 180°.$$

Proof: Prolong AC to X, making the ext. $\angle BCX$.

$$\angle BCX + \angle ACB = 2 \text{ rt. } \measuredangle s \qquad (46).$$

But $$\angle BCX = \angle A + \angle B \qquad (102).$$

$$\therefore \angle A + \angle B + \angle ACB = 2 \text{ rt. } \measuredangle s = 180° \qquad (\text{Ax. } 6).$$

Q.E.D.

105. Corollary. **The sum of any two angles of a triangle is less than two right angles.** (Ax. 5.)

106. Corollary. **A triangle cannot have more than one right angle or more than one obtuse angle.**

107. Corollary. **Two angles of every triangle are acute.**

108. Corollary. **The acute angles of a right triangle are complementary.**

Proof: Their sum = 1 rt. ∠. (104.)
Hence they are complementary. (19.)

109. Corollary. **Each angle of an equiangular triangle is 60°.**

110. Corollary. **If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.** (48.)

111. Corollary. **If two triangles have two angles of the one equal to two angles of the other, the third angle of the first is equal to the third angle of the second.** (104.)

112. Corollary. **Two triangles are congruent if a side and any two angles of the one are equal respectively to a homologous side and the two homologous angles of the other.**

Proof: The third ∠ of one △ = third ∠ of other △ (111).
∴ the △s are ≅ (76).

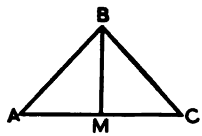
113. Corollary. **Two right triangles are congruent if a leg and the opposite acute angle of one are equal respectively to a leg and the opposite acute angle of the other.** (112.)

Ex. 1. If two angles of a triangle contain 50° and 100°, how many degrees are there in the other angle?

Ex. 2. Prove that if one angle of a triangle equals the sum of the other two angles, the triangle is a right triangle.

Ex. 3. How many degrees are there in each angle of an isosceles right triangle?

Ex. 4. The altitude of an isosceles right triangle upon the hypotenuse divides the triangle into two isosceles right triangles, each of whose acute angles is equal to 45°.



Ex 5. In a right triangle, if one acute angle is 47°, what is the other?

Ex. 6. In an isosceles triangle, if $\angle A = \angle B = 80°$, find $\angle C$.

Ex. 7. In $\triangle ABC$, if $\angle A = 25°$, $\angle B = 88°$, find $\angle C$ and the exterior angle at A.

Ex. 8. The vertex angle of an isosceles triangle is 44°. Find each base angle.

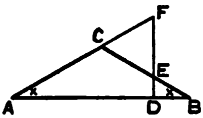
Ex. 9. If one acute angle of a right triangle is double the other, how many degrees are there in each?

Ex. 10. If one acute angle of a right triangle is five times the other, how many degrees are there in each?

Ex. 11. If any angle of an isosceles triangle is 60°, show that the triangle is equiangular.

Ex. 12. If the vertex angle of an isosceles triangle equals four times the sum of the base angles, find each angle.

Ex. 13. If the vertex angle of an isosceles triangle is twice the sum of the base angles, any line perpendicular to the base forms with the sides of the given triangle (one side to be produced) an equiangular triangle. [First find $\angle x$ and $\angle ACB$. Then use exterior $\angle s$ at D.]

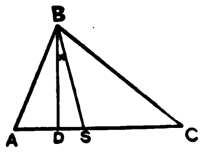


Ex. 14. The angles of a triangle are 44°, 62°, 74°. These are bisected by lines meeting at a point. Find the number of degrees in the three angles at this point.

Ex. 15. If two angles of a triangle are 80° and 55°, how many degrees are there in the angle formed by their bisectors?

Ex. 16. The vertex angle of an isosceles triangle is one third of either exterior angle at the extremities of the base. Find each angle of the triangle.

Ex. 17. If two angles of a triangle are 30° and 40°, how many degrees are there in the angle formed by the bisector of the third angle and the altitude from the same vertex? **Solution:** $\angle x = \angle ABS - \angle ABD = \frac{1}{2}\angle ABC$ − comp. of $\angle A$.



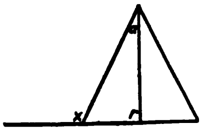
Ex. 18. The angle between the altitude of a triangle and the bisector of the angle at the same vertex equals half the difference of the other angles of the triangle.

Proof. $\angle x = ABS - \angle ABD$

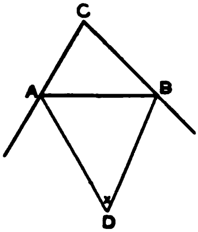
$= \frac{1}{2}(180° - \angle A - \angle C) - (90° - \angle A)$ **(Ax. 6).**

$=$ etc.

Ex. 19. The exterior angle at the base of an isosceles triangle equals half the vertex angle plus 90°.



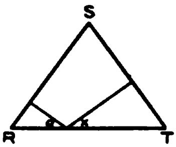
Ex. 20. If in $\triangle ABC$, $\angle BAC = 80°$, $\angle ABC = 30°$, find the angle formed by the bisectors of the exterior angles at A and B.



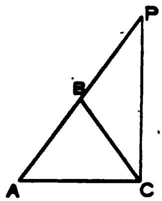
Ex. 21. The angle formed by the bisectors of two exterior angles of a triangle equals half the sum of the interior angles at the same vertices.

Ex. 22. If, in triangle ABC, the bisectors of the interior angle at B and the exterior angle at C meet at D, the angle BAC equals twice the angle BDC.

Ex. 23. The angle between the bisectors of two angles of a triangle equals half the third angle, plus a right angle.



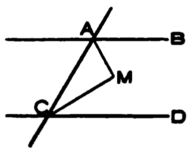
Ex. 24. If from any point in the base of an isosceles triangle perpendiculars to the equal sides are drawn, they make equal angles with the base.



Ex. 25. If one of the legs of an isosceles triangle is produced through the vertex its own length, and the extremity is joined to the nearer end of the base, this line is perpendicular to the base.

Ex. 26. If the middle point of one side of a triangle is equally distant from the three vertices, the triangle is a right triangle.

Ex. 27. If the points at which the bisectors of the equal angles of an isosceles triangle meet the opposite sides, are joined by a line, it is parallel to the base.



Ex. 28. The bisectors of two interior angles on the same side of a transversal cutting two parallels meet at right angles.

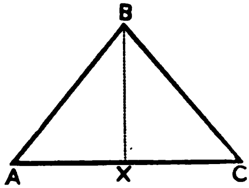
Proof: $\angle BAC + \angle ACD = 180°$ (?)

$\therefore \frac{1}{2}\angle BAC + \frac{1}{2}\angle ACD = 90°$ (Ax. 3).

That is, $\angle MAC + \angle MCA = 90°$ (Ax. 6).

$\therefore \angle M = 90°$ (104).

PROPOSITION XXXV. THEOREM



114. If two angles of a triangle are equal, the triangle is isosceles. [Converse of 55.]

Given: $\triangle ABC$; $\angle A = \angle C$.

To Prove: $AB = BC$.

Proof: Suppose BX drawn $\perp$ to AC.

In the rt. $\triangle$ ABX and CBX, $BX = BX$ (Iden.).

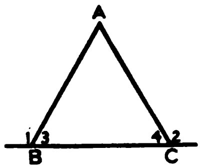
$\angle A = \angle C$ (Hyp.).

$\therefore \triangle ABX$ is congruent to $\triangle CBX$ (113).

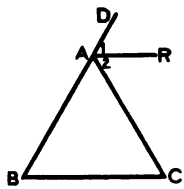
$\therefore AB = BC$ (27).

Q.E.D.

115. COROLLARY. An equiangular triangle is equilateral.

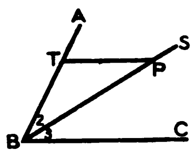


Ex. 1. If the exterior angles of a triangle are equal, the triangle is isosceles.



Ex. 2. A line parallel to the base of an isosceles triangle, meeting the equal sides, forms another isosceles triangle. Prove this for all three cases, whether it meets the equal sides, or those sides prolonged.

Ex. 3. If a line through the vertex of a triangle bisects the exterior angle and is parallel to the base, the triangle is isosceles.

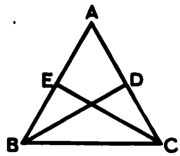


Ex. 4. If from any point in the bisector of an angle a line is drawn parallel to either side of the angle, an isosceles triangle is formed.

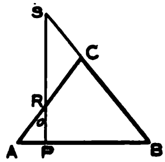
Ex. 5. The bisectors of the equal angles of an isosceles triangle form, with the base, another isosceles triangle.

Ex. 6. If from any point in the base of an isosceles triangle a line is drawn parallel to one of the equal sides and meeting the other side, an isosceles triangle is formed.

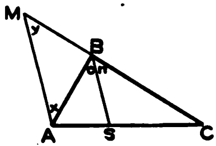
Ex. 7. If two altitudes of a triangle are equal, the triangle is isosceles. (Prove two ways.)



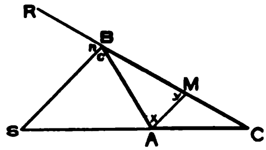
Ex. 8. If a perpendicular is erected at any point in the base of an isosceles triangle, meeting one leg, and the other leg produced, another isosceles triangle is formed.



Ex. 9. If ABC is a triangle, BS is the bisector of $\angle ABC$, and AM is parallel to BS, meeting BC produced, at M, the triangle ABM is isosceles.



Ex. 10. If a line is drawn perpendicular to the bisector of an angle and intersecting the sides, an isosceles triangle is formed.

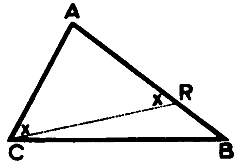


Ex. 11. If ABC is a triangle and BS is the bisector of exterior $\angle ABR$ and AM is $\parallel$ to BS, meeting BC at M, $\triangle ABM$ is isosceles.

Historical Note. Euclid of Alexandria lived during the third century B.C.; but little is known of his parentage, teachers, residence, or career. He was probably a contemporary of Eratosthenes and Archimedes. He wrote "The Elements," the most complete treatise on geometry that appeared before modern times, and inasmuch as this supplanted all others he must have made a great advance over the work of his predecessors. He was both a compiler and a discoverer. When a king of Egypt asked him about the possibility of mastering geometry with ease, Euclid is said to have replied, "There is no royal road to geometry."

EUCLID

PROPOSITION XXXVI. THEOREM



116. If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter side.

Given: $\triangle ABC$; $AB > AC$.

To Prove: $\angle ACB > \angle B$.

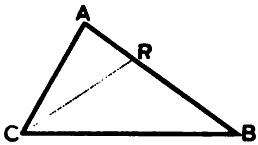
Proof: On AB take $AR =$ to AC.

Draw CR and let $\angle ARC = x$.

	$\angle ARC$ is an ext. $\angle$ of $\triangle CBR$	(Def. 101).
	$\therefore \angle x > \angle B$	(103).
But	$\angle ACR = \angle x$	(55).
Substituting,	$\angle ACR > \angle B$	(Ax. 6).
Again,	$\angle ACB > \angle ACR$	(Ax. 5).
	$\therefore \angle ACB > \angle B$	(Ax. 11).

Q.E.D.

PROPOSITION XXXVII. THEOREM



117. If two angles of a triangle are unequal, the side opposite the greater angle is longer than the side opposite the less angle. [Converse.]

Given: $\triangle ABC$; $\angle ACB > \angle B$.

To Prove: $AB > AC$.

Proof: In $\angle ACB$, suppose $\angle BCR$ constructed $=$ to $\angle B$.

Then	$CR = BR$	(114).
Also	$AR + CR > AC$	(Ax. 12).
Substituting,	$AR + BR > AC$	(Ax. 6).
That is,	$AB > AC$.	Q.E.D.

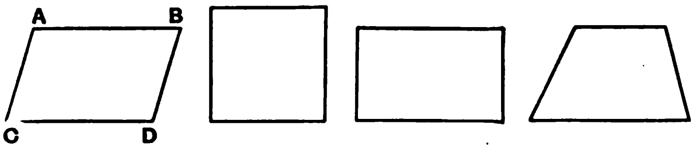
118. COROLLARY. The hypotenuse is the longest side of a right triangle. (117.)

QUADRILATERALS

119. A **quadrilateral** is a portion of a plane bounded by four straight lines. These four lines are called the **sides.** The **vertices** of a quadrilateral are the four points at which the sides intersect. The **angles** of a quadrilateral are the four angles at the vertices. The **diagonal** of a rectilinear figure is a line joining two vertices, not in the same side.

120. A **trapezium** is a quadrilateral having no two sides parallel. A **trapezoid** is a quadrilateral having two and only two sides parallel. A **parallelogram** is a quadrilateral having its opposite sides parallel (▱).

NOTE. In the first figure below, angles A, D, or B, C are **opposite** angles, angles A, C, or B, D, or A, B, or C, D are **consecutive** angles.



PARALLELOGRAM RHOMBOID — SQUARE — RECTANGLE — TRAPEZOID

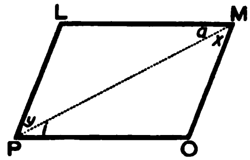
121. A **rectangle** is a parallelogram whose angles are right angles. A **rhomboid** is a parallelogram whose angles are not right angles.

122. A **square** is an equilateral rectangle. A **rhombus** is an equilateral rhomboid.

123. The side upon which a figure appears to stand is called its **base.** A trapezoid and all parallelograms have **two bases,** — the actual base and the side parallel to it. The non-parallel sides of a trapezoid are sometimes called the **legs.** An **isosceles trapezoid** is a trapezoid whose legs are equal. The **median** of a trapezoid is the line connecting the midpoints of the legs. The **altitude** of a trapezoid and of all parallelograms is the perpendicular distance between the bases.

PROPOSITION XXXVIII. THEOREM

124. The opposite sides of a parallelogram are equal.



Given: $\square$ *LMOP*.

To Prove: $LM = PO$ and $LP = MO$.

Proof: Draw diagonal *PM*.

In $\triangle$s *LMP* and *OMP*, $PM = PM$ (Iden.).

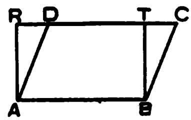
$\angle a = \angle i$ (66).

$\angle y = \angle x$ (?).

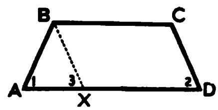
$\therefore \triangle LMP$ is congruent to $\triangle OMP$ (76).

$\therefore LM = PO$ and $LP = MO$ (27).

Q.E.D.

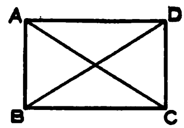


Ex. 1. If two perpendiculars are drawn to the upper base of a parallelogram from the extremities of the lower base, two congruent right triangles are formed.



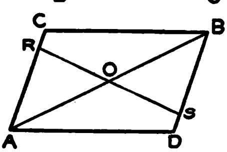
Ex. 2. The angles adjoining each base of an isosceles trapezoid are equal.

Ex. 3. If the angles at the base of a trapezoid are equal, the figure is isosceles.



Ex. 4. The diagonals of a rectangle are equal.

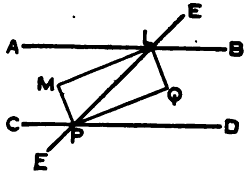
Ex. 5. If the diagonals of a parallelogram are equal, the figure is a rectangle.



Ex. 6. Any line terminated in a pair of opposite sides of a parallelogram and passing through the midpoint of a diagonal is bisected by this point.

Ex. 7. If one angle of a parallelogram is a right angle, the figure is a rectangle.

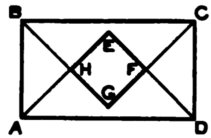
Ex. 8. The bisectors of the angles of a trapezoid form a quadrilateral, two of whose angles are right angles.



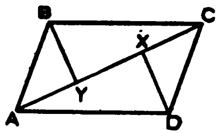
Ex. 9. The bisectors of the four interior angles formed by a transversal cutting two parallels form a rectangle.

[Prove each ∠ of $LMPQ$ a rt. ∠.]

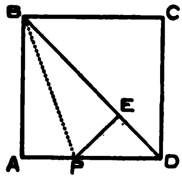
Ex. 10. The bisectors of the angles of a parallelogram form a rectangle.



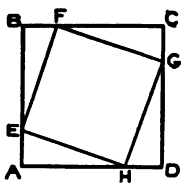
Ex. 11. The bisectors of the angles of a rectangle form a square. [In order to prove $EFGH$ equilateral, the ▲ AHB and CDF are proved congruent and isosceles; similarly ▲ BGC and AED.]



Ex. 12. The perpendiculars upon a diagonal of a parallelogram from the opposite vertices are equal.



Ex. 13. If on diagonal BD, of square $ABCD$, BE is taken equal to a side of the square, and EP is drawn perpendicular to BD meeting AD at P, $AP = PE = ED$.



Ex. 14. If $ABCD$ is a square and E, F, G, H are points on the sides, such that $AE = BF = CG = DH$, $EFGH$ is a square.

[First, prove $EFGH$ equilateral; then one ∠ a rt. ∠.]

Ex. 15. The diagonals of an isosceles trapezoid are equal.

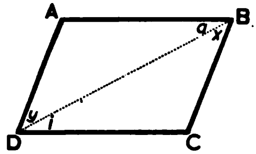
125. Corollary. **Parallel lines included between parallel lines are equal.** (124.)

126. Corollary. **The diagonal of a parallelogram divides it into two congruent triangles.**

127. Corollary. **The opposite angles of a parallelogram are equal.** (27.)

Proposition XXXIX. Theorem

128. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram. [Converse of 124.]



Given: Quadrilateral $ABCD$; $AB = DC$; $AD = BC$.

To Prove: $ABCD$ is a ▱.

Proof: Draw diagonal BD.

In $\triangle$s ABD and CBD, $BD = BD$ (Iden.).

$AB = DC$ and $AD = BC$ (Hyp.).

$\therefore \triangle ABD$ is congruent to $\triangle CBD$ (78).

$\therefore \angle a = \angle i$ (27).

$\therefore AB$ is $\parallel$ to DC (70).

Also $\angle y = \angle x$ (27).

$\therefore AD$ is $\parallel$ to BC (70).

Hence $ABCD$ is a ▱ (Def.). Q.E.D.

Proposition XL. Theorem

129. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

Given: Quadrilateral $ABCD$; $AB = DC$ and $AB \parallel$ to DC.

To Prove: $ABCD$ is a ▱.

Proof: Draw diagonal BD.

In $\triangle$s ABD and CBD, $BD = BD$ (Iden.).

$AB = DC$ (Given).

$\angle a = \angle i$ (66).

$\therefore \triangle ABD$ is congruent to $\triangle CBD$ (52).

Hence $\angle y = \angle x$ (27).

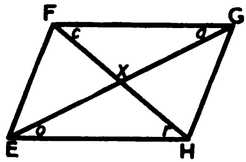
$\therefore AD$ is $\parallel$ to BC (70).

$\therefore ABCD$ is a ▱ (Def.). Q.E.D.

130. Corollary. Any pair of consecutive angles of a parallelogram are supplementary. (69.)

Proposition XLI. Theorem

131. The diagonals of a parallelogram bisect each other.



Given: ▱ *EFGH*; diagonals *EG* and *FH* intersecting at *X*.

To Prove: $FX = XH$ and $GX = XE$.

Proof: In ▵ *FXG* and *EXH*, $FG = EH$ (124).

$\angle a = \angle o$ and $\angle c = \angle r$ (66).

∴ △ *FXG* is congruent to △ *EXH* (76).

∴ $FX = XH$ and $GX = XE$ (27). Q.E.D.

Proposition XLII. Theorem

132. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Given: Quadrilateral *EFGH*, diagonals *EG* and *FH*, $FX = XH$, $EX = GX$.

To Prove: *EFGH* is a ▱.

Proof: In ▵ *FXG* and *EXH*, $FX = XH$ and $EX = XG$ (Hyp.).

$\angle FXG = \angle EXH$ (?).

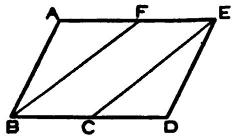
∴ △ *FXG* is congruent to △ *EXH* (?).

∴ $FG = EH$ and $\angle c = \angle r$ (?).

∴ *FG* is ∥ to *EH* (70).

∴ *EFGH* is a ▱ (129). Q.E.D.

Ex. 1. The lines joining a pair of opposite vertices of a parallelogram to the midpoints of the opposite sides are equal and parallel.

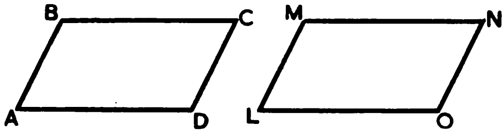


Ex. 2. If through the point of intersection of the diagonals of a parallelogram, two lines are drawn intersecting a pair of opposite sides (produced if necessary), the intercepts on these sides are equal.

Ex. 3. It is impossible to draw two straight lines from the ends of the base of a triangle terminating in the opposite sides, so that they shall bisect each other.

PROPOSITION XLIII. THEOREM

133. Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.



Given: ▱s *AC* and *LN*; $AB = LM$; $AD = LO$; $\angle A = \angle L$.

To Prove: The ▱s are congruent.

Proof: Superpose ▱ *AC* upon ▱ *LN*, so that the equal ∠s *A* and *L* coincide, *AD* falling along *LO* and *AB* along *LM*.

Point *D* coincides with point *O* [$AD = LO$ (Hyp.)].

Point *B* coincides with point *M* [$AB = LM$ (Hyp.)].

BC and *MN* are both ∥ to *LO* (Def.).

∴ *BC* falls along *MN* (Ax. 13).

CD and *NO* are both ∥ to *LM* (Def.).

∴ *CD* falls along *NO* (?).

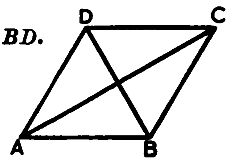
Hence *C* falls exactly upon *N* (38).

∴ the figures coincide, and are congruent (26). Q.E.D.

134. COROLLARY. Two rectangles are congruent if the base and the altitude of one are equal respectively to the base and the altitude of the other.

PROPOSITION XLIV. THEOREM

135. The diagonals of a rhombus (or of a square) are perpendicular to each other, bisect each other, and bisect the angles of the rhombus (or of the square).



Given: Rhombus *ABCD*; diagonals *AC*, *BD*.

To Prove: *AC* ⊥ to *BD*; *AC* and *BD* bisect each other; and they bisect ∠s *DAB*, *ABC*, etc.

Proof: Point A is equally distant from B and D (122).
Point C is equally distant from B and D (?).

$\therefore AC$ is $\perp$ to BD (83). Q.E.D.

Also AC and BD bisect each other (131). Q.E.D.

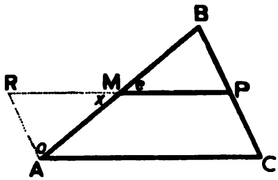
Also DB bisects $\angle ADC$ and $\angle ABC$ (85).

Similarly, AC bisects $\angle DAB$ and $\angle DCB$ (85).

Q.E.D.

Proposition XLV. Theorem

136. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.



Given: $\triangle ABC$; M, the midpoint of AB; P, the midpoint of BC; line MP.

To Prove: $MP \parallel$ to AC;
and $MP = \frac{1}{2} AC$.

Proof: Suppose AR is drawn through A, $\parallel$ to BC and meeting MP produced, at R.

In $\triangle$ ARM and BPM, $AM = BM$ (Hyp.).

$\angle x = \angle e$ (51).

$\angle o = \angle B$ (66).

$\therefore \triangle ARM$ is congruent to $\triangle BPM$ (76).

$\therefore AR = BP$ (27).

But $BP = PC$ (Hyp.).

$\therefore AR = PC$ (Ax. 1).

$\therefore ACPR$ is a $\square$ (129).

Hence RP, or MP, is $\parallel$ to AC (Def.). Q.E.D.

Also $RP = AC$ (124).

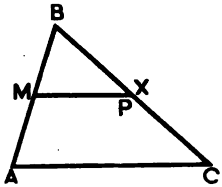
But $MP = RM$ (27).

$\therefore MP$, the half of RP, $= \frac{1}{2} AC$ (Ax. 6).

Q.E.D.

Proposition XLVI. Theorem

137. The line bisecting one side of a triangle and parallel to a second side, bisects also the third side.

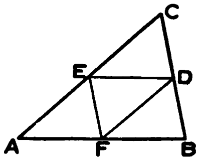


Given: $\triangle ABC$; MP bisecting AB and $\parallel$ to AC.

To Prove: MP bisects BC also.

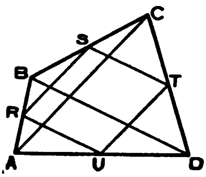
Proof: Suppose MX is drawn from M, the midpoint of AB to X, the midpoint of BC.

	MX is $\parallel$ to AC	(136).
But	MP is $\parallel$ to AC	(Hyp.).
	$\therefore MX$ and MP coincide	(Ax. 13).
	That is, MP bisects BC.	Q.E.D.



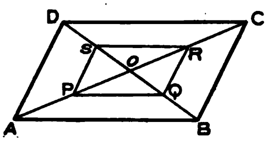
Ex. 1. The lines joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles.

Ex. 2. The lines joining (in order) the midpoints of the sides of a quadrilateral form a parallelogram the sum of whose sides is equal to the sum of the diagonals of the quadrilateral.



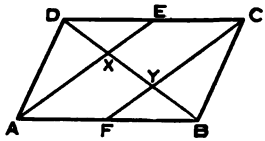
Ex. 3. The lines joining (in order) the midpoints of the sides of a rectangle form a rhombus. [Draw the diagonals.]

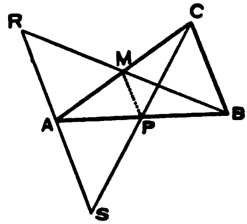
Ex. 4. If the four midpoints of the four halves of the diagonals of a parallelogram are joined in order, another parallelogram is formed.



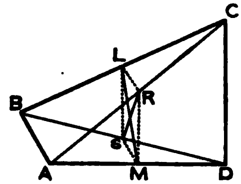
Prove this in four ways.

Ex. 5. If lines are drawn from a pair of opposite vertices of a parallelogram to the midpoints of a pair of opposite sides, they trisect the diagonal joining the other two vertices.





Ex. 6. If two medians are drawn from two vertices of a triangle and produced their own length beyond the opposite sides, and if these extremities are joined to the third vertex, these two lines are equal, and in the same straight line.

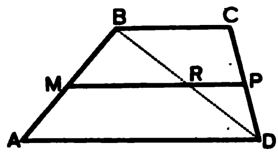


Ex. 7. The line joining the midpoints of one pair of opposite sides of a quadrilateral and the line joining the midpoints of the diagonals bisect each other.

Proposition XLVII. Theorem

138. The line bisecting one leg of a trapezoid and parallel to the base bisects the other leg, is the median, and is equal to half the sum of the bases.

Given: Trapezoid $ABCD$; M, the midpoint of AB; $MP \parallel$ to AD, meeting CD at P.



To Prove:

I. P is the midpoint of CD.

II. MP is the median.

III. $MP = \frac{1}{2}(AD + BC)$.

Proof: I. Draw diagonal BD, meeting MP at R.

MP is $\parallel$ to BC (63).

In $\triangle ABD$, MR bisects BD (137).

In $\triangle BDC$, RP bisects CD (?).

That is, P is the midpoint of CD. Q.E.D.

II. MP is the median (Def. 123). Q.E.D.

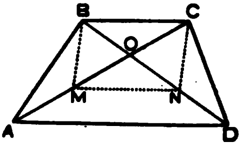
III. In $\triangle ABD$, $MR = \frac{1}{2} AD$ (136).

Also, in $\triangle BDC$, $RP = \frac{1}{2} BC$ (?).

Adding, $\therefore MP = \frac{1}{2}(AD + BC)$ (Ax. 2).

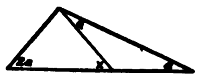
Q.E.D.

139. Corollary. **The median of a trapezoid is parallel to the bases and equal to half their sum.**

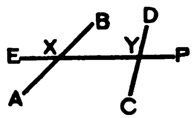


Ex. 1. In a trapezoid one of whose bases is double the other, the diagonals intersect at a point two thirds of the distance from each end of the longer base to the opposite vertex.

Proof: Take M, the midpoint of AO (136), etc.

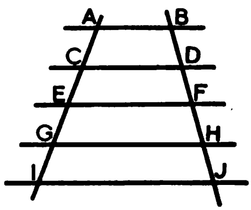


Ex. 2. If one angle of a triangle is double another, the line from the third vertex, making with the longer adjacent side an angle equal to the less given angle, divides the triangle into two isosceles triangles.



Note. The verb "*to intersect*" means merely "*to cut*." In geometry, the verb "*to intercept*" means "*to include between*." Thus the statement "AB and CD intercept XY on the line EF" really means, "AB and CD intersect EF and include XY, a part of EF, between them."

Proposition XLVIII. Theorem



140. Parallels intercepting equal parts on one transversal intercept equal parts on any transversal.

Given: ‖s AB, CD, EF, GH, IJ intercepting equal parts AC, CE, EG, GI, on the transversal AI, and cutting transversal BJ.

To Prove: $BD = DF = FH = HJ$.

Proof: The figure $ABFE$ is a trapezoid (?).

CD bisects AE and is ‖ to EF (Hyp.).

$\therefore$ D is midpoint of BF (?).

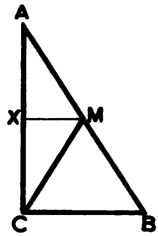
That is, $BD = DF$.

Similarly, $CDHG$ is a trapezoid and $DF = FH$.

Similarly, $FH = HJ$. $\therefore$ $BD = DF = FH = HJ$ (Ax. 1).

Q.E.D.

PROPOSITION XLIX. THEOREM



141. The midpoint of the hypotenuse of a right triangle is equally distant from the three vertices.

Given: Rt. $\triangle ABC$; M, the midpoint of the hypotenuse AB.

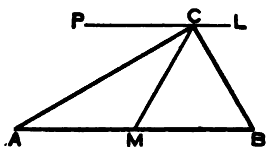
To Prove: $AM = CM = BM$.

Proof: Suppose MX drawn $\parallel$ to BC, meeting AC at X.

	X is the midpoint of AC	(137).
	MX is $\perp$ to AC	(64).
	$\therefore AM = CM$	(80).
But	$AM = BM$	(Hyp.).
	$\therefore AM = CM = BM$	(Ax. 1).

Q.E.D.

Ex. 1. Any right triangle can be divided by one line into two isosceles triangles.

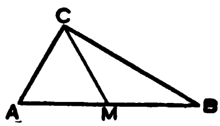


Ex. 2. If through the vertex of the right angle of a right triangle a line is drawn parallel to the hypotenuse, the legs of the right triangle bisect the angles formed by this parallel and the median drawn to the hypotenuse.

Ex. 3. If one leg of a trapezoid is perpendicular to the bases, the midpoint of the other leg is equally distant from the ends of the first leg. [Draw the median.]

Ex. 4. The median of a trapezoid bisects both the diagonals.

Ex. 5. The line joining the midpoints of the diagonals of a trapezoid is a part of the median, is parallel to the bases, and is equal to half their difference.



Ex. 6. If the median of a triangle is equal to half the side to which it is drawn, it is a right triangle.

Ex. 7. The line (prolonged if necessary) joining the midpoints of two sides of a triangle, bisects the altitude drawn to the third side.

Ex. 8. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.

Proof: Use fig. of 141. Denote $\angle A$ by x,

then $\angle B = 2x$ and $x = 30°$ (?).

$AM = MC = BM$ (141).

$\therefore \angle BCM = \angle B = 60°$ (55).

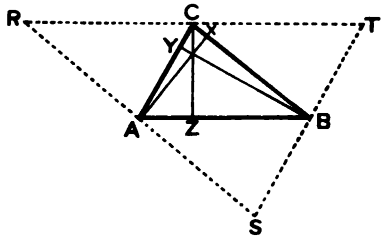
$\therefore \angle BMC = 60°$ (104).

$\therefore MB = BC$ (115).

$\therefore AB$, or $2 \times MB$, $= 2 \times CB$ (Ax. 6).

PROPOSITION L. THEOREM

142. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.



Given: $\triangle ABC$, $AX \perp$ to BC, $BY \perp$ to AC, and $CZ \perp$ to AB.

To Prove: These three $\perp$s meet in a point.

Proof: Through A suppose RS drawn $\parallel$ to BC; through B, $TS \parallel$ to AC; through C, $RT \parallel$ to AB, forming $\triangle RST$.

The figure $ABCR$ is a $\square$ (Const.).

and $ABTC$ is a $\square$ (?).

$RC = AB$ and $CT = AB$ (124).

$\therefore RC = CT$ (Ax. 1).

Now CZ is $\perp$ to RT (64).

That is, CZ is $\perp$ bisector of RT.

Similarly, AX is $\perp$ bisector of RS.

And BY is $\perp$ bisector of TS.

$\therefore$ in $\triangle RST$, AX, BY, CZ meet at a point (100).

Q.E.D.

PROPOSITION LI. THEOREM

143. The three medians of a triangle meet in a point which is two thirds the distance from any vertex to the midpoint of the opposite side.

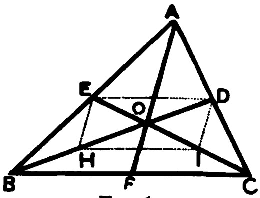


FIG. 1.

Given: $\triangle ABC$, medians AF, BD and CE, the latter two meeting at O. (Fig. 1.)

To Prove: $BO = \frac{2}{3} BD$; $CO = \frac{2}{3} CE$; $AO = \frac{2}{3} AF$ and that all three medians meet at O.

Proof: Suppose H is midpoint of BO and I is the midpoint of CO. Draw ED, DI, IH, HE.

In $\triangle ABC$, ED is $\parallel$ to BC and $= \frac{1}{2} BC$ (136).

In $\triangle OBC$, HI is $\parallel$ to BC and $= \frac{1}{2} BC$ (136).

$\therefore ED = HI$ (Ax. 1); and ED is $\parallel$ to HI (63).

$\therefore EDIH$ is a $\square$ (129).

$\therefore HO = OD$ and $IO = OE$ (131).

$\therefore BH = HO = OD$ and $CI = IO = OE$ (Ax. 1).

That is, $BO = \frac{2}{3} BD$ and $CO = \frac{2}{3} CE$.

Suppose AF meets BD at O'. (Fig. 2.)

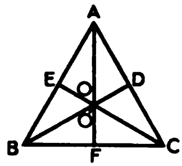


FIG. 2.

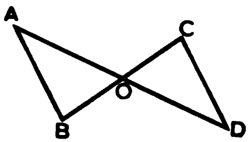
Then $BO = \frac{2}{3} BD$ (Proved above).

And $BO' = \frac{2}{3} BD$ (Proved similarly).

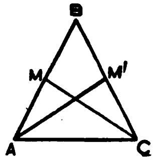
$\therefore BO = BO'$ (Ax. 1).

That is, O and O' are the same point, and the three medians meet at O, which is $\frac{2}{3}$ the distance from any vertex to the midpoint of the opposite side. Q.E.D.

Ex. 1. If two lines (AB and CD) are equal and parallel, the lines connecting their opposite ends bisect each other.

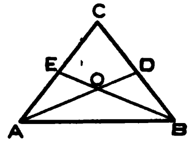


Ex. 2. If the bisector of one angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.



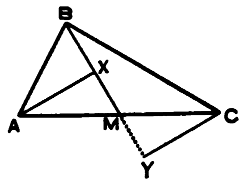
Ex. 3. The median of an isosceles triangle is perpendicular to the base.

Ex. 4. The medians from the ends of the base of an isosceles triangle are equal.



Ex. 5. If a triangle has two equal medians, it is isosceles.

Ex. 6. If ABC is an equilateral triangle and D, E, F are points on the sides, such that $AD = BE = CF$, triangle DEF is also equilateral.



Ex. 7. Any two vertices of a triangle are equally distant from the median from the third vertex.

Ex. 8. The lines bisecting two interior angles that a transversal makes with one of two parallels cut off equal segments on the other parallel from the point at which the transversal meets it. [The ▲ formed are isosceles.]

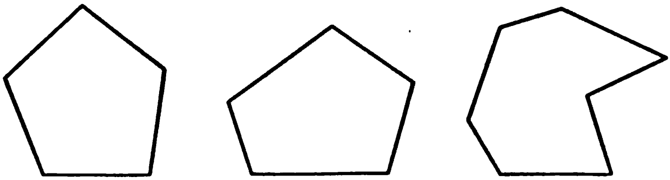
POLYGONS

144. A **polygon** is a portion of a plane bounded by straight lines. The lines are called the **sides.** The points of intersection of the sides are the **vertices.** The **angles** of a polygon are the angles at the vertices.

145. The **number** of sides of a polygon is the same as the number of its vertices or the number of its angles. An **exterior angle** of a polygon is an angle without the polygon, between one side of the polygon and another side prolonged.

146. An **equilateral polygon** has all its sides equal to one another. An **equiangular polygon** has all its angles equal to one another.

147. A **convex polygon** is a polygon no side of which if produced will enter the surface bounded by the sides of the polygon. A **concave polygon** is a polygon at least two sides of which if produced will enter the polygon.



EQUILATERAL EQUIANGULAR CONCAVE, OR
CONVEX POLYGONS REËNTRANT

NOTE. A polygon may be equilateral and not be equiangular; or it may be equiangular and not be equilateral. The word "polygon" usually signifies *convex* figures.

148. Two polygons are **mutually equiangular** if for every angle of the one there is an equal angle in the other and similarly placed. Two polygons are **mutually equilateral**, if for every side of the one there is an equal side in the other, and similarly placed.

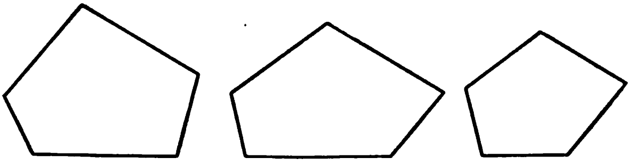
149. **Homologous angles** in two mutually equiangular polygons are the pairs of equal angles. **Homologous sides** in two polygons are the sides between two pairs of homologous angles.

150. Two **polygons** are **congruent** if they are mutually equiangular and their homologous sides are equal; or if they are composed of triangles, equal each to each and similarly placed. (Because in either case the polygons can be made to coincide.)

Ex. 1. If a quadrilateral has three equal sides, is it necessarily a parallelogram? a trapezoid?

Ex. 2. Two quadrilaterals are congruent if three sides and the two included angles of one are equal respectively to three corresponding sides and the two included angles of the other.

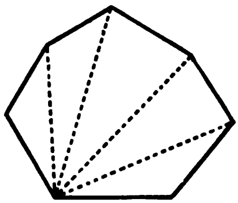
151. Two polygons may be mutually equiangular without being mutually equilateral; also, they may be mutually equilateral without being mutually equiangular — except in the case of triangles.



The first two figures are mutually equilateral, but *not* mutually equiangular. The last two figures are mutually equiangular, but *not* mutually equilateral.

152. A 3-sided polygon is a triangle.
A 4-sided polygon is a quadrilateral.
A 5-sided polygon is a pentagon.
A 6-sided polygon is a hexagon.
A 7-sided polygon is a heptagon.
An 8-sided polygon is an octagon.
A 10-sided polygon is a decagon.
A 12-sided polygon is a dodecagon.
A 15-sided polygon is a pentadecagon.
An n-sided polygon is called an n-gon.

Proposition LII. Theorem



153. The sum of the interior angles of an n-gon is equal to $(n-2)$ times 180°.

Given: A polygon having n sides.

To Prove: The sum of its interior $\angle s = (n-2) \cdot 180°$.

Proof: If all possible diagonals are drawn from any vertex it is evident that there are formed $(n-2)$ triangles.

The sum of the $\angle s$ of one $\triangle = 180°$ (104).

$\therefore$ the sum of the $\angle$s of $(n-2)$ $\triangle$s $= (n-2)\ 180°$ (Ax. 3).
The sum of $\angle$s of the $\triangle$s $=$ sum of $\angle$s of n-gon (Ax. 4).
$\therefore$ the sum of $\angle$s of the n-gon $= (n-2) \cdot 180°$ (Ax. 1). Q.E.D.

154. Corollary. **The sum of the interior angles of an n-gon is equal to $180° \cdot n - 360°$.**

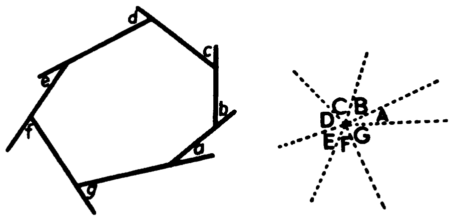
155. Corollary. **Each angle of an equiangular n-gon $= \dfrac{(n-2)\ 180°}{n}$.**

156. Corollary. **The sum of the angles of any quadrilateral is equal to four right angles.**

157. Corollary. **If three angles of a quadrilateral are right angles, the figure is a rectangle.**

Proposition LIII. Theorem

158. **If the sides of a polygon are produced in order, one at each vertex, the sum of the exterior angles of the polygon equals four right angles, that is, 360°.**



Given: A polygon with sides prolonged in succession forming the several exterior angles a, b, c, d, etc.

To Prove: $\angle a + \angle b + \angle c + \angle d +$ etc. $= 4$ rt. $\angle$s $= 360°$.

Proof: Suppose at any point in the plane, lines are drawn parallel to the several sides of the given polygon, extending in the same direction, and forming $\angle$s A, B, C, D, etc.

Then $\angle A + \angle B + \angle C + \angle D + \angle E +$ etc. $= 4$ rt. $\angle$s (47).
But $\angle A = \angle a$, $\angle B = \angle b$, $\angle C = \angle c$, $\angle D = \angle d$, etc. (74).
Substituting,

$\angle a + \angle b + \angle c + \angle d +$ etc. $= 4$ rt. $\angle$s $= 360°$ (Ax. 6).

Q.E.D.

159. Corollary. **Each exterior angle of an equiangular polygon is equal to $\frac{4 \text{ rt. } \angle s}{n}$, that is, to $\frac{360^\circ}{n}$.**

160. Corollary. **The sum of the exterior angles of a polygon is independent of the number of its sides.**

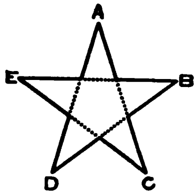
Ex. 1. How many degrees are there in the sum of all the angles of a pentagon? of a decagon? of a dodecagon?

Ex. 2. How many degrees are there in each angle of an equiangular hexagon? of an equiangular octagon?

Ex. 3. How many degrees are there in each exterior angle of an equiangular pentagon? of an equiangular hexagon? 16-gon?

Ex. 4. How many sides has the polygon the sum of whose interior angles exceeds the sum of its exterior angles by 900°?

Ex. 5. The sum of the angles at the vertices of a five-pointed star (pentagram) is equal to two right angles.



Ex. 6. If two angles of a quadrilateral are supplementary, the other two are supplementary.

Ex. 7. If from any point within an angle perpendiculars to the sides are drawn, they include an angle which is the supplement of the given angle.

Ex. 8. If a quadrilateral has four equal sides, what kind is it?

Ex. 9. Can a trapezoid have three equal sides? a trapezium?

Ex. 10. A colt is tied, by a chain 30 ft. long, to the four corners of a lot 60 ft. square, on four successive days. Using a scale of $\frac{1}{8}$ in. to the foot, draw a diagram showing the area over which the colt grazes during these four days. Draw another diagram, using the same scale, for a chain 15 ft. long; 40 ft. long.

Ex. 11. A calf is tied by a chain 20 ft. long, to the four corners of a barn 40 ft. square, on four successive days and allowed to graze in the adjoining pasture. Using a scale of $\frac{1}{8}$ in. to the foot, draw a diagram showing the area grazed over at the end of the fourth day. Draw another diagram, using the same scale, for a chain 15 ft. long; 25 ft. long.

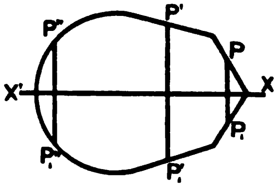
SYMMETRY

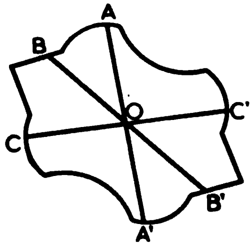
161. A figure is **symmetrical** with respect to a **line** if, by using that line as an axis, the part of the figure on one side of the line may be folded over, and will exactly coincide with the part on the other side. This line is an **axis of symmetry.**

162. A figure is **symmetrical** with respect to a **point** if this point bisects every line drawn through it and terminated (both ways) in the boundary of the figure.

This point is the **center of symmetry.**

163. It is evident that the axis of symmetry bisects at right angles every line joining two symmetrical points; and that the center of symmetry bisects every line joining any pair of points symmetrical with respect to it.





Examples of symmetry are given in these figures.

First figure is symmetrical with respect to XX' as an axis. (Why?)

Second figure is symmetrical with respect to O as a center. (Why?)

P and P_1' are symmetrical with respect to XX' as an axis. (Why?)

A and A', B and B', etc. are symmetrical with respect to O as a center. (Why?) XX' is $\perp$ to $P'P_1'$ and bisects it. $AO = A'O$, $BO = B'O$, etc.

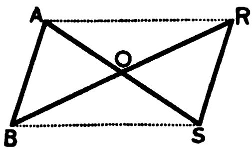
Ex. 1. Is the altitude of an isosceles triangle an axis of symmetry? the altitude of a scalene triangle?

Ex. 2. Is the diagonal of a rhomboid an axis of symmetry? of a square?

Ex. 3. Has any triangle a center of symmetry? a parallelogram?

Proposition LIV. Theorem

164. If two lines are symmetrical with respect to a center, they are equal and parallel.



Given: AB and RS symmetrical with respect to O, that is, every line through O, terminated in AB and RS, is bisected at O; AS and BR, two such lines.

To Prove: $AB = RS$ and $AB \parallel$ to RS.

Proof: Draw AR and BS. $AO = OS$ and $BO = OR$ (Hyp.).

$\therefore$ $ABSR$ is a $\square$ (132).

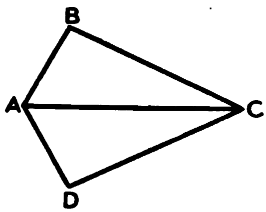
$\therefore$ $AB = RS$ (124).

Also AB is $\parallel$ to RS (Def. 120).

Q.E.D.

Proposition LV. Theorem

165. If a diagonal of a quadrilateral bisects two of its angles, this diagonal is an axis of symmetry.



Given: Quadrilateral $ABCD$; AC a diagonal bisecting $\angle BAD$ and $\angle BCD$.

To Prove: $ABCD$ symmetrical with respect to AC.

Proof: In $\triangle$s ABC and ADC, $AC = AC$ (?).

$\angle BAC = \angle DAC$ and $\angle BCA = \angle DCA$ (?).

$\therefore$ $\triangle ABC$ is congruent to $\triangle ADC$ (?).

$\therefore$ AC is an axis of symmetry (161).

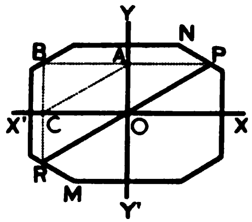
Q.E.D.

166. Corollary. The diagonal of a square or of a rhombus is an axis of symmetry.

Ex. The point of intersection of the diagonals of a parallelogram is a center of symmetry. Prove.

PROPOSITION LVI. THEOREM

167. If a figure is symmetrical with respect to two perpendicular axes, it is symmetrical with respect to their intersection as a center.



Given: Figure *MN* symmetrical with respect to the $\perp$ axes XX' and YY' which intersect at O.

To Prove: Figure *MN* is symmetrical with respect to O as a center.

Proof: Take any point P in the boundary. Draw $PB \perp$ to YY', intersecting YY' at A and meeting the boundary at B. Draw $BR \perp$ to XX', intersecting XX' at C and meeting the boundary at R. Draw AC, OP, OR.

[The demonstration is accomplished by proving POR a straight line, bisected at O.]

	PB is $\parallel$ to XX' and BR is $\parallel$ to YY'	(62).
	$\therefore ABCO$ is a $\square$	(Def.).
	$\therefore BC = AO$	(124).
But	$BC = CR$	(161).
	$\therefore AO = CR$	(Ax. 1).
Hence	$ACRO$ is a $\square$	(129).
	$\therefore RO = CA$	(124).
Also	RO is $\parallel$ to CA	(120).
Similarly,	$ACOP$ may be proved a $\square$.	
	$\therefore PO$ is $=$ to AC and $\parallel$ to AC.	
Hence	POR is a straight line	(Ax. 13).
And	$PO = RO$	(Ax. 1).

But P is *any* point in the boundary, so POR is *any* line through O.

$\therefore O$ is a center of symmetry (161).

Q.E.D.

Ex. Prove that no triangle can have a center of symmetry.

CONCERNING ORIGINAL EXERCISES

168. In the original work which this text contains, the pupil is expected to state the hypothesis and the conclusion of each theorem, and to apply them to an appropriate figure; also to give a complete and logical statement of the proof, with a reason for every statement.

In many of these exercises, suggestions are made and such assistance is given as experience has shown to be needed by average pupils. This is done in order to encourage definite accomplishment, which is one of the greatest incentives to further effort.

To apply the knowledge acquired from the preceding pages is now the student's task. His interest in this science will depend largely on the success of his efforts to prove originals.

The student should not draw a special figure for a general proposition. That is, if "triangle" is specified, he should draw a scalene and not an isosceles or a right triangle; and if "quadrilateral" is mentioned, he should draw a trapezium and not a parallelogram or a square.

SUMMARY. GENERAL DIRECTIONS FOR ATTACKING EXERCISES

169. A **triangle** is proved **isosceles** by showing that it contains two equal sides, or two equal angles.

170. A **triangle** is proved a **right** triangle by showing that one of its angles is a right angle, or that two of its angles are complementary, or that one of its angles is equal to the sum of the other two.

171. **Right triangles** are proved congruent by showing that they have:

(1) Hypotenuse and acute angle of one equal etc.
(2) Hypotenuse and leg of one equal etc.
(3) The legs of one equal etc.
(4) Leg and adjoining angle of one equal etc.
(5) Leg and opposite angle of one equal etc.

172. Oblique triangles are proved congruent by showing that they have :

(1) Two sides and the included angle of one equal etc.
(2) One side and the adjoining angles of one equal etc.
(3) Three sides of one equal etc.

173. Angles are proved equal by showing that they are :

(1) Equal to the same or to equal angles.
(2) Halves or doubles of equals.
(3) Vertical angles.
(4) Complements or supplements of equals.
(5) Homologous parts of congruent figures.
(6) Base angles of an isosceles triangle.
(7) Corresponding angles, alternate interior angles, etc., of parallels.
(8) Angles whose sides are respectively parallel or perpendicular.
(9) Third angles of triangles which have two angles of one equal etc.

174. Lines are proved equal by showing that they are :

(1) Equal to the same or to equal lines.
(2) Halves or doubles of equals.
(3) Distances to the ends of a line from any point in its perpendicular bisector.
(4) Homologous parts of congruent figures.
(5) Sides of an isosceles triangle.
(6) Distances to the sides of an angle from any point in its bisector.
(7) Opposite sides of a parallelogram.
(8) The parts of one diagonal of a parallelogram made by the other.

175. Two lines are proved **perpendicular** by showing that they:

(1) Make equal adjacent angles with each other.
(2) Are legs of a right triangle.
(3) Have two points in one, each equally distant from the ends of the other.

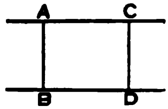
176. Two lines are proved **parallel** by:

(1) The customary angle relations of parallel lines.
(2) Showing that they are opposite sides of a parallelogram.
(3) Showing that they are parallel or perpendicular to a third line.

177. Two lines, or two angles, are proved **unequal** by the usual axioms and theorems pertaining to inequalities.

[See especially, Ax. 5; Ax. 12; 81, 86, 87, 88, III, 91, 92, 103, 116, 118.]

ORIGINAL EXERCISES



1. Parallel lines are everywhere equally distant.

Given: $\|_s$ AC and BD; AB and CD $\perp_s$ to AC.

To Prove: $AB = CD$.

2. If two lines in a plane are everywhere equally distant, they are parallel.

3. The bisectors of any two consecutive angles of a parallelogram meet at right angles.

4. The line drawn from any point in the base of an isosceles triangle to the opposite vertex is less than either leg.

5. If ABC is an equilateral triangle and each side is produced (in order) the same distance, so that $AD = BE = CF$, the triangle DEF is equilateral.

6. If $ABCD$ is a square and the sides are produced (in order) the same distance, so that $AE = BF = CG = DH$, the figure $EFGH$ is a square.

7. The two lines joining the midpoints of the opposite sides of a quadrilateral bisect each other. [Join the 4 midpoints (in order), etc.]

8. If two adjacent angles of a quadrilateral are right angles, the bisectors of the other angles are perpendicular to each other.

9. If two opposite angles of a quadrilateral are right angles, the bisectors of the other angles are parallel.

10. Two isosceles triangles are congruent, if:

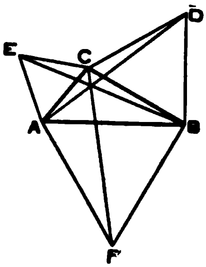
(1) The base and one of the adjoining angles in the one are equal respectively to the base and one of the adjoining angles in the other.

(2) A leg and one of the base angles in the one are equal respectively to a leg and one of the base angles in the other.

(3) The base and vertex angle in one are equal to the same in the other.

(4) A leg and vertex angle in one are equal to the same in the other.

(5) A leg and the base in one are equal to the same in the other.



11. If upon the three sides of any triangle equilateral triangles are constructed (externally) and a line is drawn from each vertex of the given triangle to the farthest vertex of the opposite equilateral triangle, these three lines are equal.

Proof: $\angle EAC = \angle BAF$ (?). Add to each of these $\angle CAB$.

$\therefore \angle EAB = \angle CAF$ (?).

Then prove $\triangle$ EAB and CAF congruent.

Similarly, $\triangle CAD$ is congruent to $\triangle CEB$. Etc.

12. The sum of the diagonals of any quadrilateral is less than the sum of the four sides, but greater than half that sum.

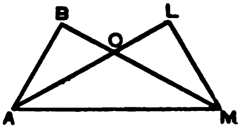
13. The difference between two sides of a triangle is less than the third side.

14. The bisectors of the exterior angles of a rectangle form a square.

15. The bisectors of the equal angles of an isosceles triangle (terminating in the equal sides) are equal.

16. The median to the base of an isosceles triangle bisects the vertex angle.

17. The perpendiculars to the legs of an isosceles triangle from the midpoint of the base are equal.



18. State and prove the converse of Ex. 17.

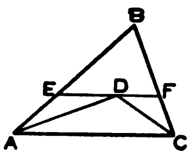
19. If $AB = LM$ and $AL = BM$, $\angle B = \angle L$ and $\angle BAO = \angle OML$ and $BO = OL$.

20. The bisectors of a pair of corresponding angles are parallel.

21. If two lines are cut by a transversal and the exterior angles on the same side of the transversal are supplementary, the lines are parallel.

22. The bisectors of a pair of vertical angles are in the same straight line.

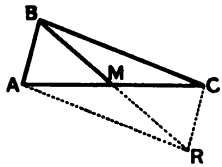
23. The midpoint of a diagonal of a parallelogram is a center of symmetry.



24. If the base angles of a triangle are bisected and through the intersection of the bisectors a line is drawn parallel to the base and terminating in the sides, this line is equal to the sum of the parts of the sides it meets, between it and the base.

25. In two congruent triangles, homologous medians are equal; homologous altitudes are equal; homologous bisectors are equal.

26. If two parallel lines are cut by a transversal, the two exterior angles on the same side of the transversal are supplementary.

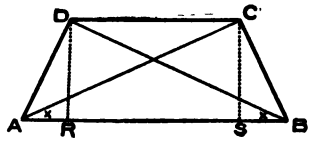


27. If from a point a perpendicular is drawn to each of two parallels, they are in the same line. [Draw a third ∥ through the point.]

28. The median to one side of a triangle is less than half the sum of the other two sides.

Proof: Produce median BM to R, so that $MR = BM$, draw AR and CR. Fig. is ▱. (?). Etc.

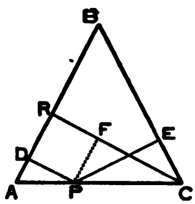
29. The sum of the medians of a triangle is less than the sum of the sides of the triangle.



30. If the diagonals of a trapezoid are equal, it is isosceles.

[Draw DR and CS ⊥ to AB; and prove rt. ▲ ACS and BDR congruent, to get $\angle x = \angle x$.]

31. If a perpendicular is drawn from each vertex of a parallelogram to any line outside the parallelogram, the sum of the ⊥s from one pair of opposite vertices equals the sum of the ⊥s from the other pair.

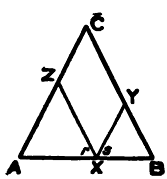


32. The sum of the perpendiculars to the legs of an isosceles triangle from any point in the base equals the altitude upon one of the legs. (That is, the sum of the perpendiculars from any point in the base of an isosceles triangle to the equal sides remains a uniform length for every point of the base.)

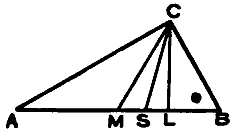
[Prove $PE = CF$.]

33. The sum of the three perpendiculars drawn from any point within an equilateral triangle, to the three sides, remains a uniform length for all positions of the point.

[Draw a line through this point ∥ to one side; draw the altitude of the △ ⊥ to this line and side; prove the sum of the three ⊥s equals this altitude and hence equals a constant.]



34. If from any point in the base of an isosceles triangle parallels to the equal sides are drawn, the sum of the sides of the parallelogram formed is equal to the sum of the legs of the triangle.

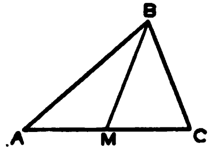


35. The bisector of the right angle of a right triangle is also the bisector of the angle formed by the median and the altitude drawn from the same vertex.

To Prove: $\angle MCS = \angle LCS$.

Proof: $\angle ACS = \angle BCS$ (?); $\angle ACM = \angle BCL$ (?). Now use Ax. 2.

36. If the vertex angle of an isosceles triangle is equal to the sum of the base angles, any line perpendicular to the base forms with the sides of the given triangle (one side to be produced) three isosceles right triangles.



37. If two sides of a triangle are unequal and the median to the third side is drawn, the angles formed with the base are unequal.

38. State and prove the converse of Ex. 37.

39. If the opposite sides of a hexagon are equal and parallel, the three diagonals drawn between opposite vertices meet in a point.

40. In triangle ABC, AD is perpendicular to BC, meeting it at D; E is the midpoint of AB, and F of AC; the angle EDF is equal to the angle EAF.

41. If the diagonals of a quadrilateral are equal, and also one pair of opposite sides, two of the four triangles into which the quadrilateral is divided by the diagonals are isosceles.

42. If angle A of triangle ABC equals three times angle B, there can be drawn a line AD meeting BC in D, such that the triangles ABD and ACD are isosceles.

43. If E is the midpoint of side BC of parallelogram $ABCD$, AE and BD meet at a point two thirds the distance from A to E and from D to B.

44. If in triangle ABC, in which AB is not equal to AC, AC' is taken on AB (produced if necessary) equal to AC, and AB' is taken on AC (produced if necessary) equal to AB, and $B'C'$ is drawn meeting BC at D, then AD bisects angle BAC.

Proof: $\triangle ABC$ is congruent to $\triangle AB'C'$ (?) (52). $\therefore$ their homologous parts are equal. Thus prove that $\triangle BC'D$ is congruent to $\triangle B'CD$. Etc.

45. If a diagonal of a parallelogram bisects one angle, it also bisects the opposite angle.

47. If a diagonal of a parallelogram bisects one angle, the figure is equilateral.

48. Any line drawn through the point of intersection of the diagonals of a parallelogram divides the figure into two congruent quadrilaterals.

49. If AR bisects angle A of triangle ABC and AT bisects the exterior angle at A, any line parallel to AB, having its extremities in AR and AT, is bisected by AC.

50. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

51. If, in isosceles triangle XYZ, AD is drawn from A, the midpoint of YZ, perpendicular to the base XZ, $DZ = \frac{1}{4} XZ$. [Draw alt. from Y.]

52. If ABC is an equilateral triangle, if the bisectors of angles B and C meet at D, if DE is drawn parallel to AB meeting AC at E, and DF, parallel to BC meeting AC at F, then $AE = ED = EF = DF = CF$.

53. If A is any point in RS of triangle RST, and B is the midpoint of RA, C the midpoint of AS, D the midpoint of ST, and E the midpoint of TR, then $BCDE$ is a parallelogram.

54. If lines are drawn from any vertex of a parallelogram to the midpoints of the two opposite sides, they divide the diagonal which they intersect into three equal parts.

Proof: Draw the other diagonal.

55. If the interior and exterior angles at two vertices of a triangle are bisected, a quadrilateral is formed, having two of its angles right angles and the other two supplementary.

56. The four bisectors of the angles of a quadrilateral form a second quadrilateral whose opposite angles are supplementary.

Proof: Extend a pair of opposite sides of the given quadrilateral to meet at X. Bisect the base angles of the new $\triangle$ formed, meeting at O. Then show that $\angle O$ equals one of the $\angle$s between the given bisectors, and $\angle O$ is supplementary to the angle opposite.

BOOK II

THE CIRCLE

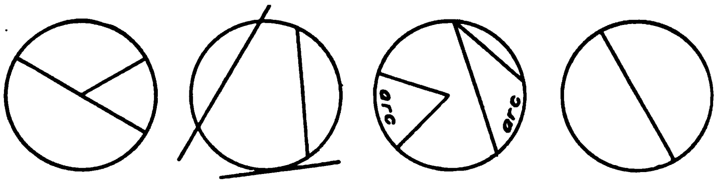
178. A **curved line** is a line no part of which is straight.

179. A **circle** is a plane curve all points of which are equally distant from a point in the plane, called the **center.**

180. The length of the circle is called the **circumference.**

181. A **radius** is a straight line drawn from the center to any point of the circle.

A **diameter** is a straight line that contains the center, and the extremities of which are in the circle.



CIRCLE
RADIUS
DIAMETER

SECANT CHORD
TANGENT
POINT OF CONTACT

CENTRAL ANGLE
INSCRIBED ANGLE
ARC

SEMI-
CIRCUMFERENCES
SEMICIRCLES

A **secant** is a straight line cutting the circle in two points.

A **chord** is a straight line the extremities of which are in the circle.

A **tangent** is a straight line which touches the circle at only one point, and does not cut it, however far it may be extended. The point at which the line touches the circle is called the **point of contact** or the **point of tangency.**

A **common tangent** to two circles is a line tangent to both of them.

182. A **central angle** is an angle formed by two radii.

An **inscribed angle** is an angle whose vertex is on the circle and whose sides are chords.

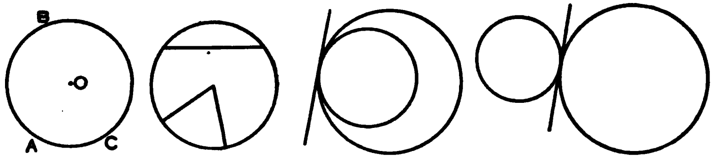
183. An **arc** is any part of a circle.

A **semicircle** is an arc equal to half a circle.

A **quadrant** is an arc equal to one fourth of a circle.

Equal circles are circles having equal radii.

Concentric circles are circles having the same center.



CIRCLE O SECTOR SEGMENT CIRCLES INTERNALLY TANGENT CIRCLES EXTERNALLY TANGENT

NOTE. A circle is named either by its center or by three of its points as "the $\odot$ *O*," or "the $\odot$ *ABC*."

184. A **sector** is the figure bounded by two radii and their included arc.

A **segment** is the figure bounded by an arc and its chord.

185. Two **circles** are **tangent** to each other if they are tangent to the same line at the same point. Circles may be tangent to each other **internally**, if the one is within the other, or **externally**, if each is without the other.

186. POSTULATE. **A circle can be described about any given point as center and with any given line as radius.**

Subtend is used in the sense of "to cut off." A chord **subtends** an arc. Hence an arc is **subtended by** a chord.

An **angle** is said to **intercept** the arc between its sides. Hence an arc is **intercepted by** an angle.

The **hypothesis** is contained in what constitutes the subject of the principal verb of the theorem.

PRELIMINARY THEOREMS

187. THEOREM. **All radii of the same circle are equal.** (179.)

188. THEOREM. **All radii of equal circles are equal.** (183.)

189. THEOREM. **The diameter of a circle equals twice the radius.**

190. THEOREM. **All diameters of the same or of equal circles are equal.** (Ax. 3.)

191. THEOREM. **The diameter of a circle bisects the circle.**

Given: Any ⊙ and a diameter.

To Prove: The diameter bisects the circle.

Proof: Suppose one segment folded over upon the other segment, using the diameter as an axis. If the arcs do not coincide, there are points of the circle unequally distant from the center. But this is impossible (179).

∴ the arcs coincide and are equal (26).

Q.E.D.

192. THEOREM. **With a given point as center and a given line as radius, it is possible to describe only one circle.** (179.)

That is, a circle is determined if its center and radius are fixed.

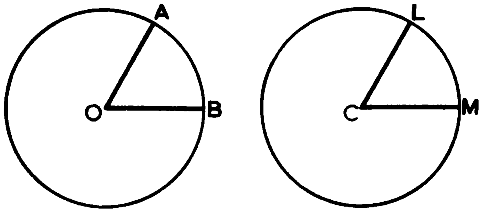
Historical Note. Archimedes was born at Syracuse, Sicily, during the third century B.C. He is regarded as the greatest mathematician of antiquity, and probably of all time, save only that modern wizard, Sir Isaac Newton. He was educated in Egypt and won the respect and admiration of the king, Hiero, for his exceptional genius in the construction of mechanical devices and mathematical formulas. These included the measurement of the circle (circumference and area), the cone, the cylinder, and the sphere. To him is given credit also for the discovery of specific gravity. Cicero relates his discovery of the tomb of Archimedes over a century after his burial in Syracuse.

ARCHIMEDES

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

193. In the same circle (or in equal circles) equal central angles intercept equal arcs.



Given: $\odot O = \odot C$; $\angle O = \angle C$.

To Prove: Arc AB = arc LM.

Proof: Superpose $\odot O$ upon the equal $\odot C$, making $\angle O$ coincide with its equal, $\angle C$. Point A falls on L, and point B on M (188).

Arc AB coincides with arc LM (179).

$\therefore$ arc AB = arc LM (26).

Q.E.D.

Proposition II. Theorem

194. In the same circle (or in equal circles) equal arcs are intercepted by equal central angles. [Converse.]

Given: $\odot O = \odot C$; arc AB = arc LM.

To Prove: $\angle O = \angle C$.

Proof: Superpose $\odot O$ upon the equal $\odot C$, making the centers coincide. Point A falls on point L. Then arc AB coincides with arc LM and point B falls on point M. (Because the arcs are equal.)

$\therefore$ OA coincides with CL, and OB with CM (39).

$\therefore \angle O = \angle C$ (26).

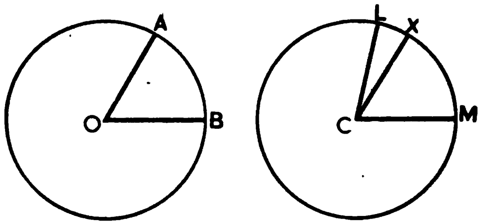
Q.E.D.

Proposition III. Theorem

195. In the same circle (or in equal circles):

I. **If two central angles are unequal, the greater angle intercepts the greater arc.**

II. **If two arcs are unequal, the greater arc is intercepted by the greater central angle.** [Converse.]



I. **Given:** $\odot O = \odot C$; $\angle LCM > \angle O$.

To Prove: Arc $LM >$ arc AB.

Proof: Superpose $\odot O$ upon $\odot C$, making sector AOB fall in position of sector XCM, OB coinciding with CM.

CX is within the angle LCM. (Because $\angle LCM > \angle O$.)

Arc AB falls upon LM, in the position XM (179).

$\therefore$ arc $LM >$ arc XM (Ax. 5).

That is, arc $LM >$ arc AB. Q.E.D.

II. **Given:** (?).

To Prove: $\angle LCM > \angle O$.

Proof: The pupil may employ either superposition, as in I, or the method of exclusion, as in 92.

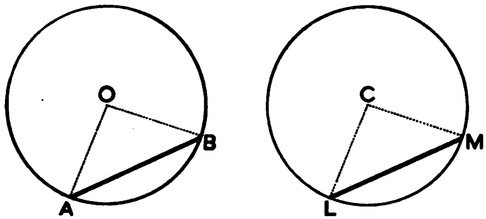
Note. Unless otherwise specified, the arc of a chord always refers to the *lesser* of the two arcs. If *two* arcs (in the same or equal circles) are concerned, it is understood either that each is less than a semicircle, or each is greater.

Ex. 1. Are equal circles also congruent? Why?

Ex. 2. Is there a geometrical figure that is both sector and segment?

PROPOSITION IV. THEOREM

196. In the same circle (or in equal circles) equal chords subtend equal arcs.



Given: $\odot O = \odot C$; chord $AB =$ chord LM.

To Prove: Arc $AB =$ arc LM.

Proof: Draw the several radii to the ends of the chords.

In $\triangle$s OAB and CLM, $OA = CL$ and $OB = CM$ (188).

Chord $AB =$ chord LM (Hyp.).

$\therefore \triangle OAB$ is congruent to $\triangle CLM$ (?).

Hence $\angle O = \angle C$ (?).

$\therefore$ arc $AB =$ arc LM (193).

Q.E.D.

PROPOSITION V. THEOREM

197. In the same circle (or in equal circles) equal arcs are subtended by equal chords. [Converse.]

Given: $\odot O = \odot C$; arc $AB =$ arc LM.

To Prove: Chord $AB =$ chord LM.

Proof: Draw the several radii to the ends of the chords.

In $\triangle$s OAB and CLM, $OA = CL$ (188).

$OB = CM$ (80).

$\angle O = \angle C$ (194).

$\therefore \triangle AOB$ is congruent to $\triangle CLM$ (?).

$\therefore$ chord $AB =$ chord LM (?).

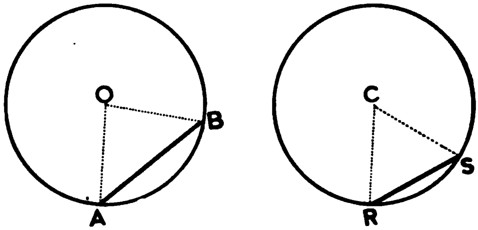
Q.E.D.

Proposition VI. Theorem

198. In the same circle (or in equal circles):

I. **If two chords are unequal, the greater chord subtends the greater arc.**

II. **If two arcs are unequal, the greater arc is subtended by the greater chord.** [Converse.]



I. **Given:** $\odot O = \odot C$; chord $AB >$ chord RS.

To Prove: Arc $AB >$ arc RS.

Proof: Draw the several radii to the ends of the chords.

In $\triangle$ AOB and RCS,

$AO = RC$ and $BO = SC$	(?).
Chord $AB >$ chord RS	(Hyp.).
$\therefore \angle O > \angle C$	(92).
$\therefore$ arc $AB >$ arc RS	(195, I).

Q.E.D.

II. **Given:** $\odot O = \odot C$; arc $AB >$ arc RS.

To Prove: Chord $AB >$ chord RS.

Proof: Draw the several radii.

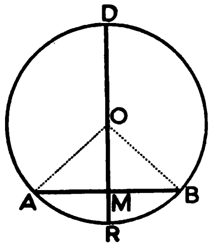
In $\triangle$ AOB and RCS,

	$AO = RC$ and $BO = SC$	(?).
But	$\angle O > \angle C$	(195, II).
	$\therefore$ chord $AB >$ chord RS	(91).

Q.E.D.

Ex. Can either part of Proposition VI be proved by the method of exclusion? Can Proposition IV or V be proved by that method?

PROPOSITION VII. THEOREM



199. The diameter perpendicular to a chord bisects the chord and both the subtended arcs.

Given: Diameter $DR \perp$ to chord AB in $\odot O$.

To Prove:

I. $AM = MB$;

II. Arc AR = arc RB, arc AD = arc DB.

Proof: Draw radii to ends of the chord.

I. In rt. $\triangle$s OAM and OBM,

$$OA = OB \qquad (?).$$

$$OM = OM \qquad (?).$$

$$\triangle OAM \text{ is congruent to } \triangle OBM \qquad (84).$$

$$\therefore AM = BM \qquad (?).$$

Q.E.D.

II. $\angle AOM = \angle BOM$ (27).

$\therefore$ arc AR = arc RB (193).

Also $\angle AOD = \angle BOD$ (49).

$\therefore$ arc AD = arc DB (193).

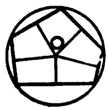
Q.E.D.

200. COROLLARY. **The line from the center of a circle perpendicular to a chord bisects the chord.**

201. COROLLARY. **The perpendicular bisector of a chord passes through the center of the circle.**

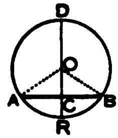
Proof: The center is equally distant from the extremities of the chord (187).

$\therefore$ the center is in the $\perp$ bisector of the chord (82).

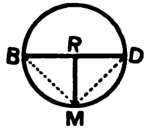


Ex. 1. The perpendicular bisectors of all chords in a circle pass through a common point.

Ex. 2. A diameter bisecting a chord is perpendicular to the chord and bisects the subtended arcs.



Ex. 3. A diameter bisecting an arc is the perpendicular bisector of the chord of the arc.



Ex. 4. A line bisecting a chord and its arc is the perpendicular bisector of the chord.

Ex. 5. If a circle is described on the hypotenuse of a right triangle as diameter, it passes through the vertex of the right angle (141).

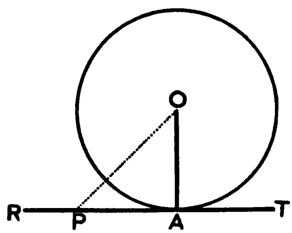
Ex. 6. If any number of parallel chords are drawn in a circle, their midpoints all lie on the same straight line.

Ex. 7. If two perpendicular diameters of a circle are drawn and their extremities are joined in order, these chords form a square.

Ex. 8. If any two diameters of a circle are drawn and their extremities are joined in order, the figure is a parallelogram.

Proposition VIII. Theorem

202. The line perpendicular to a radius at its extremity is tangent to the circle.



Given: Radius OA of ⊙ O, and $RT \perp$ to OA at A.

To Prove: RT tangent to the circle.

Proof: Take any point P in RT (except A) and draw OP.

$OP > OA$ (87).

∴ P lies without the ⊙ (Because $OP >$ radius).

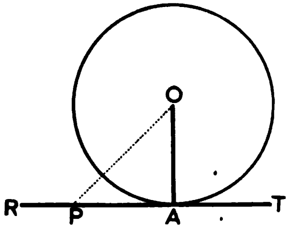
That is, every point (except A) in RT is without the ⊙.

∴ RT is a tangent to the ⊙ O (Def.).

Q.E.D.

Proposition IX. Theorem

203. If a line is tangent to a circle, the radius drawn to the point of contact is perpendicular to the tangent. [Converse.]



Given: RT tangent to $\odot O$ at A; radius OA.

To Prove: $OA \perp$ to RT.

Proof: Every point (except A) in RT is without the $\odot$ (181).

$\therefore$ a line from O to any point in RT (except A) is $> OA$. (Because it is $>$ a radius.)

That is, OA is the shortest line from O to RT.

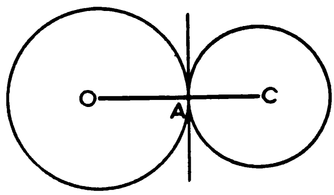
$\therefore OA$ is $\perp$ to RT (87).

204. Corollary. **The perpendicular to a tangent at the point of contact passes through the center of the circle.** (43.)

Q.E.D.

Proposition X. Theorem

205. If two circles are tangent to each other, the line joining their centers passes through their point of contact.



Given: ⓢ O and C tangent to a line at A, and line OC.

To Prove: OC passes through A.

Proof: Draw radii OA and CA.

OA is $\perp$ to the tangent and CA is $\perp$ to the tangent (203).

$\therefore OAC$ is a straight line (43).

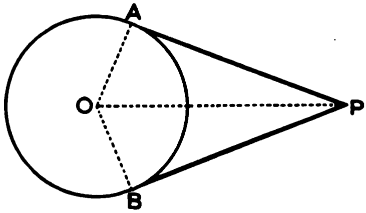
$\therefore OAC$ and OC coincide, and OC passes through A (39).

Q.E.D.

Let the pupil supply the proof if the circles are tangent *internally*.

Proposition XI. Theorem

206. Two tangents drawn to a circle from an external point are equal.



Note. In this theorem the word "tangent" signifies the distance between the external point and the point of contact.

Given: $\odot O$ and tangents PA, PB.

To Prove: Distance PA = distance PB.

Proof: Draw radii to points of contact, and join OP.

$\angle OAP$ and OBP are right $\angle$s (203).

In rt. $\triangle$s OAP and OBP, $OP = OP$ (?); $OA = OB$ (?).

$\therefore \triangle OAP$ is congruent to $\triangle OBP$ (84).

$\therefore PA = PB$ (?).

Q.E.D.

Historical Note. Pythagoras, a Greek philosopher, born probably at Samos, in the sixth century B.C., had the reputation of being an "assiduous inquirer," and of having a great fund of general knowledge. He was a moral reformer as well as a scientific teacher. He was the head of a secret society the members of which were pledged to the severest discipline, and to the practice of temperance, purity, and obedience. The study of mathematics in Greece was magnified by him and advanced to the rank of a science. The invincible proof given on page 204 of the theorem that the square on the hypotenuse of a right triangle is equal in area to the sum of the squares on the legs, has been attributed to Pythagoras, and is often referred to as the Pythagorean proposition. The discovery of incommensurable magnitudes (p. 96) and of many other theorems and problems is ascribed to him.

Pythagoras

Proposition XII. Theorem

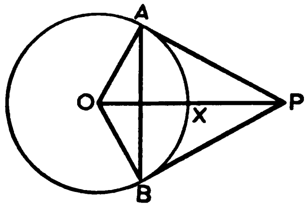
207. If from an external point tangents are drawn to a circle, and radii are drawn to the points of contact, the line joining the center and the external point bisects:

I. The angle formed by the tangents.

II. The angle formed by the radii.

III. The chord joining the points of contact.

IV. The arc intercepted by the tangents.



Given: Tangents AP and BP from point P and radii OA and OB.

To Prove: Line OP bisects:

I. $\angle APB$, II. $\angle AOB$, III. Chord AB, IV. Arc AXB.

Proof: $\triangle$ OAP and OBP are rt. $\triangle$ (?).

They are congruent. (Explain.)

I. $\therefore \angle APO = \angle BPO$ (?).

II. $\angle AOP = \angle BOP$ (?).

III. O is equidistant from A and B (?).

P is also equidistant from A and B (206).

$\therefore OP$ is $\perp$ to AB at its midpoint (83).

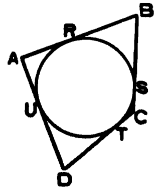
IV. Arc AX = arc BX (193).

Q.E.D.

Ex. 1. Tangents drawn to a circle at the extremities of a diameter are parallel.

Ex. 2. Tangents drawn to a circle at the extremities of a chord form, with the chord, an isosceles triangle.

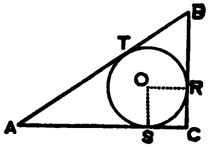
Ex. 3. The bisector of the angle between two tangents to a circle passes through the center.



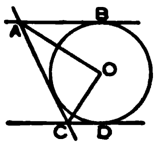
Ex. 4. The sum of one pair of opposite sides of a circumscribed quadrilateral is equal to the sum of the other pair.

Ex. 5. A circumscribed parallelogram is equilateral.

Ex. 6. A circumscribed rectangle is a square.

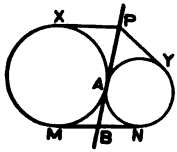


Ex. 7. If a circle is inscribed in a right triangle, the sum of the diameter and the hypotenuse is equal to the sum of the legs.



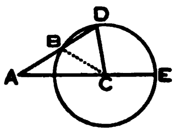
Ex. 8. If two parallel tangents meet a third tangent, and lines are drawn from the points of intersection to the center, they are perpendicular.

Ex. 9. Tangents drawn to two tangent circles from any point in their common interior tangent are equal.



Ex. 10. The common interior tangent of two tangent circles bisects their common exterior tangent.

Ex. 11. Do the theorems of Ex. 9 and 10 apply if the circles are tangent internally? If so, prove.



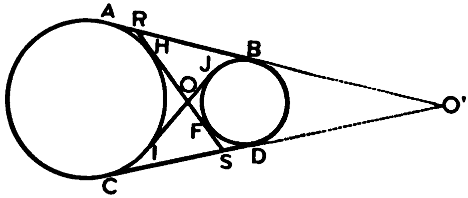
Ex. 12. In the adjoining figure, if AE and AD are secants, AE passing through the center, and the external part of AD being equal to a radius, the angle $DCE = 3\angle A$.

[Draw BC. $\angle DBC =$ ext. $\angle$ of $\triangle ABC = 2\angle A = \angle D$. (Explain.) $\angle DCE =$ an ext. $\angle$, etc.]

Ex. 13. The two common interior tangents of two circles are equal.

Ex. 14. The common exterior tangents to two circles are equal.

[Produce them to intersection.]



Ex. 15. In the preceding figure, prove that $RH = SF$.

Proof: $AR + RB = CS + SD$;

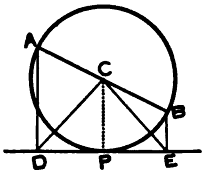
$\therefore AR + (RH + HF) = (SF + HF) + SD.$

$\therefore RH + RH + HF = SF + HF + SF$; $\therefore 2\,RH = 2\,SF$, etc. Give reasons and explain.

Ex. 16. The common exterior tangents to two circles intercept on a common interior tangent (produced), a line equal to a common exterior tangent. **To Prove:** $RS = AB$.

Ex. 17. AB and AC are two tangents from A; in the less arc BC a point D is taken and a tangent drawn at D, meeting AB at E and AC at F. Prove that $AE + EF + AF$ remains a uniform length for all positions of D in arc BC.

Ex. 18. If perpendiculars are drawn upon a tangent from the ends of any diameter:



(1) The point of tangency bisects the line between the feet of the perpendiculars.

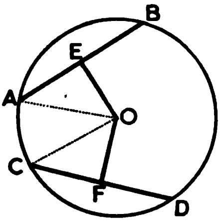
[Draw CP.]

(2) The sum of the perpendiculars equals the diameter.

(3) The center of the circle is equally distant from the feet of the perpendiculars.

PROPOSITION XIII. THEOREM

208. In the same circle (or in equal circles) equal chords are equally distant from the center.



Given: $\odot O$; chord $AB =$ chord CD, and distances OE and OF.

To Prove: $OE = OF$.

Proof: Draw radii OA and OC.

In the rt. $\triangle$ AOE and COF,

	$AE = \frac{1}{2} AB$ and $CF = \frac{1}{2} CD$	(200).
But	$AB = CD$	(Hyp.).
	$\therefore AE = CF$	(Ax. 3).
Also	$AO = CO$	(?).
	$\therefore \triangle AOE$ is congruent to $\triangle COF$	(84).
	$\therefore OE = OF$	(?).

Q.E.D.

Proposition XIV. Theorem

209. In the same circle (or in equal circles) chords which are equally distant from the center are equal. [Converse.]

Given: $\odot O$; chords AB and CD; distance $OE =$ distance OF.

To Prove: Chord $AB =$ chord CD.

Proof: Draw radii OA and OC.

In rt. $\triangle$ AOE and COF, $AO = CO$ (?).

Also $EO = OF$ (Hyp.).

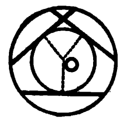
$\therefore \triangle AOE$ is congruent to $\triangle COF$ (84).

$\therefore AE = CF$ (?).

Now AB is twice AE and CD is twice CF (200).

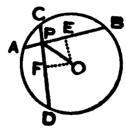
$\therefore AB = CD$ (Ax. 3). Q.E.D.

Ex. 1. If two circles are concentric, all chords of the greater that are tangent to the less are equal.



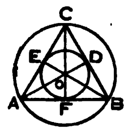
Ex. 2. If at the midpoint of an arc a tangent is drawn, it is parallel to the chord of the arc.

Ex. 3. If two equal chords intersect on the circle, the radius drawn to their point of intersection bisects their angle.



Ex. 4. If the line joining the point of intersection of two chords and the center bisects the angle formed by the chords, they are equal.

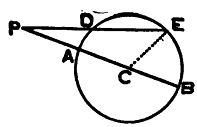
Ex. 5. The radius of the circle inscribed in an equilateral triangle is half the radius of the circle circumscribed about it. [Use 143.]



Ex. 6. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

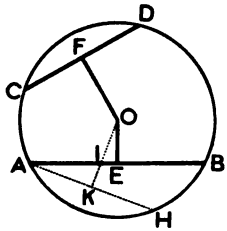
Ex. 7. If two circles are concentric and a secant cuts them both, the portions of the secant intercepted between the circumferences are equal.

Ex. 8. Of all secants that can be drawn to a circumference from a fixed external point, the longest passes through the center.



Ex. 9. The shortest line from an external point to a circumference is that which, if produced, would pass through the center.

Proposition XV. Theorem



210. In the same circle (or in equal circles) if two chords are unequal, the greater chord is at the less distance from the center.

Given: ⊙ O; chord $AB >$ chord CD, and distances OE and OF.

To Prove: $OE < OF$.

Proof: Arc $AB >$ arc CD (198, I).

Suppose arc AH is taken on arc AB, equal to arc CD. Draw chord AH. Draw $OK \perp$ to AH, cutting AB at I.

Now	chord $AH =$ chord CD	(197).
	$\therefore$ distance $OK =$ distance OF	(208).
But	$OE < OI$	(87).
Also	$OI < OK$	(Ax. 5).
	$\therefore OE < OK$	(Ax. 11).
Substituting,	$OE < OF$	(Ax. 6).

Q.E.D.

Proposition XVI. Theorem

211. In the same circle (or in equal circles) if two chords are unequally distant from the center, the chord at the less distance is the greater. [Converse.]

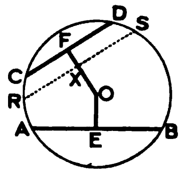
Given: ⊙ O; chords AB and CD; distance $OE <$ distance OF.

To Prove: Chord $AB >$ chord CD.

Proof: It is evident that chord $AB < CD$, or $=$ chord CD, or $>$ chord CD. Proceed by the method of exclusion.

Another Proof: On OF take $OX =$ to OE. At X draw a chord $RS \perp$ to OX.

Then	chord RS is $\parallel$ to chord CD.	(62).
	$\therefore$ arc $RS >$ arc CD	(Ax. 5).



∴ chord $RS >$ chord CD (198, II).

But chord $AB =$ chord RS (209).

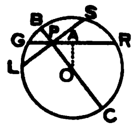
Substituting,

chord $AB >$ chord CD (Ax. 6).

Q.E.D.

212. Corollary. **The diameter of a circle is longer than any other chord.**

Ex. 1. What is the longest chord that can be drawn through a given point within a circle?



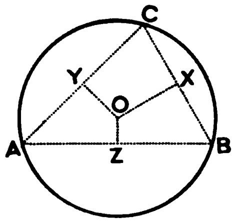
Ex. 2. Of all chords that can be drawn through a given point within a circle, the chord perpendicular to the diameter through the given point is the shortest.

Given: *P*, the point; *BOC* the diam.; *LS* ⊥ to *BC* at *P*; *GR*, any other chord through *P*.

To Prove: (?)

Proof: Draw *OA* ⊥ to *GR*. Etc.

Proposition XVII. Theorem



213. Through three points, not in the same straight line, one circle can be drawn, and only one.

Given: Points A and B and C.

To Prove: I. (?). II. (?).

Proof: I. Draw lines AB, BC, AC. Suppose their ⊥ bisectors, OZ, OX, OY, are drawn. These ⊥s will meet at a point (100).

With O as a center and OA or OB or OC as a radius, a circle can be described through A, B, and C (100).

II. These ⊥s meet at only one point (100).

That is, there is only one center.

The distances OA, OB, OC are all equal (100).

That is, there is only one radius.

∴ there can be only one circle (192). Q.E.D.

214. Corollary. **One circle, and only one, can be drawn through the vertices of a triangle.**

215. Corollary. **A circle is determined by three points.**

216. Corollary. **A circle cannot be drawn through three points which are in the same straight line.**

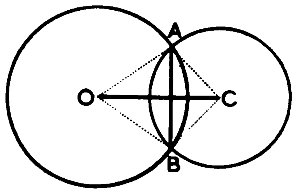
[The ⊥s would be ∥.]

217. Corollary. **A straight line can intersect a circle in only two points.** (216.)

218. Corollary. **Two circles can intersect in only two points.**

Proposition XVIII. Theorem

219. If two circles intersect, the line joining their centers is the perpendicular bisector of their common chord.



Given: (?).

To Prove: (?).

Proof: Draw radii in each ⊙ to ends of AB.

Point O is equally distant from A and B (187).

Point C is equally distant from A and B (?).

∴ OC is the ⊥ bisector of AB (83).

Q.E.D.

Ex. 1. Illustrate the five corollaries on this page by diagrams.

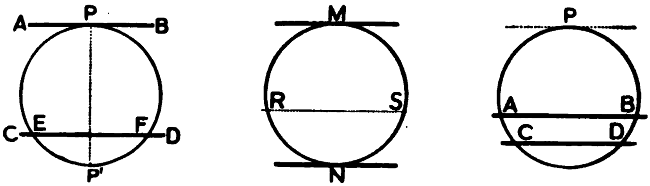
Ex. 2. On an island six miles from the mainland is a gun having a range of ten miles. Draw a diagram, using a scale of ¼ in. to the mile, showing the range of the gun.

Ex. 3. On the opposite sides of the entrance to a harbor are two forts, twelve miles apart. In each there is a gun with a range of nine miles. Draw a diagram showing the region exposed to the fire of each gun, and to the fire of both guns.

Ex. 4. Make a similar problem using three forts, and guns of different ranges, and draw the diagram, showing regions exposed to one gun only and to all three guns (if any).

PROPOSITION XIX. THEOREM

220. Parallel lines intercept equal arcs on a circle.



Given: A circle and a pair of parallels intercepting two arcs.

To Prove: The intercepted arcs are equal.

There may be three cases:

I. If the ‖s are a tangent (AB, tangent at P) and a secant (CD, cutting the circle at E and F).

Proof: Draw diameter to point of contact, P.

This diameter is $\perp$ to AB (203).

PP' is also $\perp$ to EF (64).

$\therefore$ arc $EP =$ arc FP (199).

II. If the ‖s are two tangents, points of contact being M and N.

Proof: Suppose a secant is drawn ‖ to one of the tangents, cutting the $\odot$ at R and S.

RS will be ‖ to the other tangent (63).

$\therefore$ arc $MR =$ arc MS and arc $RN =$ arc SN (I).

Adding, arc $MRN =$ arc MSN (Ax. 2).

III. If the ‖s are two secants, one cutting the $\odot$ at A and B, the other at C and D.

Proof: Suppose a tangent is drawn, ‖ to AB and touching the $\odot$ at P. This tangent is ‖ to CD (63).

Arc $PC =$ arc PD and arc $PA =$ arc PB (I).

Subtracting, arc $AC =$ arc BD (Ax. 2).

Q.E.D.

221. A **polygon** is **inscribed** in a circle, or a **circle** is **circumscribed** about a polygon if the vertices of the polygon are in the circle, and its sides are chords.

A **polygon** is **circumscribed** about a circle, or a **circle** is **inscribed** in a polygon if the sides of the polygon are all tangent to the circle.

The **perimeter** of a figure is the sum of all its bounding lines.

Exercises in Drawing Circles

1. Draw two unequal intersecting circles. Show that the line joining their centers is less than the sum of their radii.

2. Draw two circles externally (not tangent) and show that the line joining their centers is greater than the sum of their radii.

3. Draw two circles tangent externally. Discuss these lines similarly.

4. Draw two circles tangent internally. Discuss these lines similarly.

5. Draw two circles so that they can have only one common tangent.

6. Draw two circles so that they can have two common tangents.

7. Draw two circles so that they can have three common tangents.

8. Draw two circles so that they can have four common tangents.

9. Draw two circles so that they can have no common tangent.

SUMMARY

222. The following is a summary of the truths relating to magnitudes, which have been already established in Book II.

I. **Arcs are equal** if they are:

(1) Intercepted by equal central angles.
(2) Subtended by equal chords.
(3) Intercepted by parallel lines.
(4) Halves of the same arc, or of equal arcs.

II. **Lines are equal** if they are:

(1) Radii of the same or of equal circles.
(2) Diameters of the same or of equal circles.
(3) Chords that subtend equal arcs.
(4) Chords that are equally distant from the center.
(5) Tangents to one circle from the same point.

III. **Unequal arcs** and **unequal chords** have like relations.

ORIGINAL EXERCISES

1. Show that an inscribed trapezoid is isosceles.

2. In the figure of 242, show that arc DBX = arc DB + arc AC.

3. In the figure of 243, show that arc CX = arc CMB − arc CNB.

4. In the figure of 244, show that arc CX = arc CE − arc BD.

5. In the figure of 245, show that arc BX = arc BE − arc BD.

6. Show that the perpendiculars to the sides of a circumscribed polygon at the points of contact meet at a common point.

7. Show that the bisectors of the angles of a circumscribed polygon meet at a common point.

8. If two circles intersect and the four radii are drawn to the points of intersection, prove that the line joining the centers of the circles bisects the central angles formed by these radii.

9. If two chords of a circle are equal, but not parallel, and their midpoints are joined by a line, prove that the line from the center of the circle to the midpoint of the other line is perpendicular to it.

10. If a hexagon is circumscribed about a circle, prove that the sum of three alternate sides equals the sum of the other three sides.

11. Draw two circles which can have neither a common chord nor a common tangent.

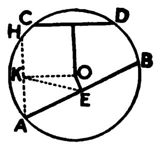
12. Two perpendicular radii are prolonged to meet a tangent to a circle, and from the two points of intersection two other tangents are drawn to this circle. Prove that these two tangents are parallel.

(Hint. Draw radii to the three points of contact.)

13. If from the midpoint of an arc perpendiculars are drawn to the radii drawn to the ends of the arc, prove that these perpendiculars are equal.

14. If through the extremities of a diameter two equal chords are drawn, one on each side of the diameter, prove that they are parallel.

15. Prove the theorem of 210 by the accompanying figure.



[Hint, in $\triangle AEK$ show that two sides are unequal, hence two ∠s are unequal, hence two angles in $\triangle EKO$ are unequal, etc.]

16. Prove the theorem of 211 by this figure, and a method similar to that employed in Ex. 15.

KINDS OF QUANTITIES — MEASUREMENT

223. A **ratio** is the quotient of one quantity divided by another — both being of the same kind.

224. To **measure a quantity** is to find the number of times it contains another quantity of the same kind, called the **unit.** This *number* is the ratio of the quantity to the unit.

225. Two quantities are called **commensurable** if there exists a common unit of measure which is contained in each a whole (integral) number of times.

Two quantities are called **incommensurable** if there does not exist a common unit of measure which is contained in each a whole number of times.

Thus, \$17 and \$35 are commensurable, but \$17 and \$$\sqrt{35}$ are incommensurable. Two lines $18\frac{1}{4}$ ft. and 13 yd. are commensurable, but $18\frac{1}{4}$ ft. and $\sqrt[3]{13}$ yd. are incommensurable.

226. A **constant** quantity is a quantity the value of which does not change during a discussion. A constant may have only one value.

A **variable** is a quantity that has different successive values during a discussion. It may have an unlimited number of values.

227. The **limit of a variable** is a constant, *to which* the variable cannot be equal, but *from which* the variable can be made to differ by less than any mentionable quantity.

228. Illustrative. The *ratio* of 15 yd. to 25 yd. is written either $\frac{15}{25}$ or $15 \div 25$ and is equal to three fifths. If we state that a son is two thirds as old as his father, we mean that the son's age divided by the father's, equals two thirds. A ratio is a fraction.

The statement that a certain distance is 400 yd. signifies that the unit (the yard), if applied to this distance, will be contained exactly 400 times.

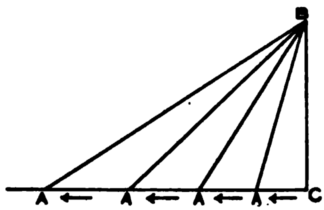
Are \$7.50 and \$3.58 *commensurable* if the unit is \$1? 1 dime? 1 cent?

Are 10 ft. and $\sqrt{19}$ ft. *commensurable?*

The height of a steeple is a *constant;* the length of its shadow made by the sun is a *variable.* The distance a train goes *varies* with the time it travels. Our ages are *variables.* The length of a standard yard, mile,

or meter, etc., is a *constant.* The height of a growing plant or a child is a *variable.*

The *limit* of a variable may be illustrated by considering a right triangle ABC, and supposing the vertex A to move farther and farther from the vertex of the right angle. It is evident that the hypotenuse becomes longer, that AC increases, but BC remains the same length. The angle A decreases, the angle B increases, but the angle C remains constantly a right angle. If we carry vertex A toward the left indefinitely, the $\angle A$ becomes less and less but cannot become zero. [Because then there could be no Δ.]



Hence the limit of the decreasing $\angle A$ is zero.

Likewise, the $\angle B$ becomes larger and larger but cannot become equal to a right angle. [Because then two sides of the triangle would be parallel, which is impossible.] But it may be made as nearly equal to a right angle as we choose.

Hence the limit of $\angle B$ is a right angle.

To these limits we cannot make the variables equal, but *from* these limits we can make them differ by less than any mentionable angle, however small.

The following supplies another illustration of the limit of a variable. The sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$ etc., will always be less than 2, no matter how many terms are collected. But by taking more and more terms we can make the actual difference between this sum and 2 less than any conceivable fraction, however small. Hence 2 is the limit of the sum of the series. The limit is not 3 or 4, because the difference between the sum and 3 cannot be made less than any assigned fraction. Neither is the limit $1\frac{1}{2}$. (Why not?) Similarly, the limit of the value of .333333 ····· *ad infinitum* is $\frac{1}{3}$.

Certain variables become equal to a fixed magnitude; but this fixed magnitude is not a limit. Thus, the length of the shadow of a tower really becomes equal to a fixed distance (at noon). A man's age really attains to a definite number of years and then ceases to vary (at death).

Hence if a variable approaches a constant, and the difference between the two can be made indefinitely small while the variable cannot become equal to the constant, the constant is the **limit of the variable.** This is merely another definition of a limit.

Proposition XX. Theorem of Limits

229. If two variables are always equal and each approaches a limit, their limits are equal.

Given: Two variables v and v'; v always $= v'$; also v approaching the limit l; v' approaching the limit l'.

To Prove: $l = l'$.

Proof: v is always $=$ to v' (Hyp.). Hence they may be considered as a single variable. Now a single variable can approach only one limit (228). Hence $l = l'$. Q.E.D.

230. (1) **Algebraic principles concerning variables.**

If v is a variable and k is a constant:

I. $v + k$ is a variable.

II. $v - k$ is a variable.

III. $k \pm v$ is a variable.

IV. kv is a variable.

V. $\frac{v}{k}$ is a variable.

VI. $\frac{k}{v}$ is a variable.

These six statements are obvious.

(2) **Algebraic principles concerning limits.**

If v is a variable whose limit is l, and k is a constant:

I. $v \pm k$ will approach $l \pm k$ as a limit.

II. $k \pm v$ will approach $k \pm l$ as a limit.

III. kv will approach kl as a limit.

IV. $\frac{v}{k}$ will approach $\frac{l}{k}$ as a limit.

V. $\frac{k}{v}$ will approach $\frac{k}{l}$ as a limit.

Note. A variable, as applied to Plane Geometry, is not added to, subtracted from, multiplied by, or divided by another variable.

Proofs: I. v cannot $= l$ (227). $\therefore v \pm k$ cannot $= l \pm k$.
Also, $v - l$ approaches zero (227).
$\therefore (v \pm k) - (l \pm k)$ approaches zero. (Because it reduces to $v - l$.)
Hence $v \pm k$ approaches $l \pm k$ (227).

II. Demonstrated similarly.

III. If $kv = kl$, then $v = l$ (Ax. 3). But this is impossible (227).
$\therefore kv$ cannot $= kl$.

Also $v - l$ approaches zero (227).

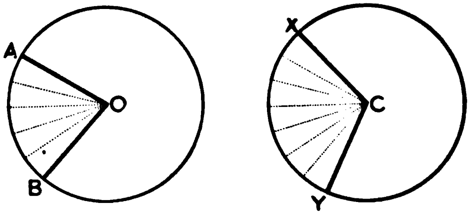
$\therefore k(v - l)$ or $kv - kl$ approaches zero.

Therefore kv approaches kl (227).

IV and V. Demonstrated similarly.

Proposition XXI. Theorem

231. In the same circle (or in equal circles) the ratio of two central angles is equal to the ratio of their intercepted arcs.



Given: $\odot O = \odot C$; central $\angle$s O and C; arcs AB and XY.

To Prove: $\dfrac{\angle O}{\angle C} = \dfrac{\text{arc } AB}{\text{arc } XY}$.

Proof: I. If the arcs are **commensurable.** There exists a common unit of measure of AB and XY (225).

Suppose this unit, when applied to the arcs, is contained 5 times in AB and 7 times in XY.

$$\therefore \frac{\text{arc } AB}{\text{arc } XY} = \frac{5}{7} \qquad \text{(Ax. 3).}$$

Draw radii to the several points of division of the arcs. $\angle O$ is divided into 5 parts and $\angle C$ into 7 parts.

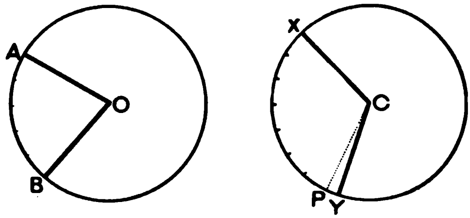
These 12 parts are all equal (194).

$$\therefore \frac{\angle O}{\angle C} = \frac{5}{7} \qquad \text{(Ax. 3).}$$

$$\therefore \frac{\angle O}{\angle C} = \frac{\text{arc } AB}{\text{arc } XY} \qquad \text{(Ax. 1).}$$

Q.E.D.

II. If the arcs are **incommensurable**. There does not exist a common unit (225). Suppose arc *AB* is divided into equal parts (any number of them). Apply one of these as a unit of measure to arc *XY*. There is a remainder *PY*. (Because *AB* and *XY* are incommensurable.)



Draw *CP*. Now $\frac{\angle O}{\angle XCP} = \frac{\text{arc } AB}{\text{arc } XP}$ (Case I).

Indefinitely increase the number of subdivisions of arc *AB*. Then each part, that is, our unit or divisor, is indefinitely decreased. Hence *PY*, the remainder, is indefinitely decreased. (Because the remainder < the divisor.)

That is, arc *PY* approaches zero as a limit.

and $\angle PCY$ approaches zero as a limit.

$\therefore$ arc *XP* approaches arc *XY* as a limit (227).

and $\angle XCP$ approaches $\angle XCY$ as a limit (227).

$\therefore \frac{\angle O}{\angle XCP}$ approaches $\frac{\angle O}{\angle XCY}$ as a limit

and $\frac{\text{arc } AB}{\text{arc } XP}$ approaches $\frac{\text{arc } AB}{\text{arc } XY}$ as a limit.

$$\therefore \frac{\angle O}{\angle XCY} = \frac{\text{arc } AB}{\text{arc } XY} \quad (229).$$

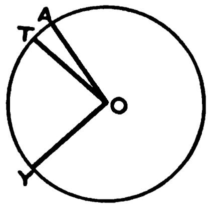
Q.E.D.

Ex. 1. If you double an arc do you double its central angle? its chord?

Ex. 2. If in two equal circles, an arc in one is taken three times as long as an arc in the other, how do their central angles compare? Is there any similar law that you know, applying to their chords?

Ex. 3. Two arcs of a circle contain 80° and 120° respectively. What is the ratio of their central angles?

PROPOSITION XXII. THEOREM



232. A central angle is measured by its intercepted arc.

Given: $\odot O$; $\angle AOY$; arc AY.

To Prove: $\angle AOY$ is measured by the arc AY, that is, they contain the same number of units.

Proof: The sum of all $\angle$s about $O = 4$ rt. $\angle$s $= 360°$ (47).

If this $\odot$ is divided into 360 equal parts and radii are drawn to the several points of division, there will be 360 equal central $\angle$s (194).

Each of these 360 central angles will be a *degree of angle* (20).

Each of the 360 equal arcs is called a *degree of arc*. Take $\angle AOT$, one of these degrees of angle, and arc AT, one of the degrees of arc. Then $\dfrac{\angle AOY}{\angle AOT} = \dfrac{\text{arc } AY}{\text{arc } AT}$ (231).

$\dfrac{\angle AOY}{\angle AOT} = \angle AOY \div$ a unit $=$ the number of units in $\angle AOY$ (224).

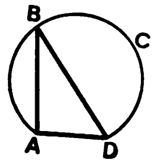
$\dfrac{\text{arc } AY}{\text{arc } AT} =$ arc $AY \div$ a unit $=$ the number of units in arc AY (224).

$\therefore$ the number of units in $\angle AOY =$ the number of units in arc AY (Ax. 1).

That is, $\angle AOY$ is measured by arc AY.

Q.E.D.

233. COROLLARY. **A central right angle intercepts a quadrant of arc.** (Because each contains 90 units.)

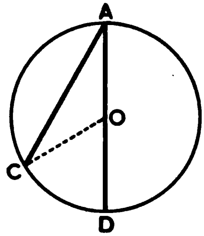


234. COROLLARY. **A right angle is measured by half a semicircle, that is, by a quadrant.**

235. An **angle** is **inscribed** in a segment if its vertex is on the arc and its sides are drawn to the ends of the arc of the segment.

Thus, $ABCD$ is a segment and $\angle ABD$ is inscribed in it.

PROPOSITION XXIII. THEOREM



236. An inscribed angle is measured by half its intercepted arc.

Given: $\odot O$; inscribed $\angle A$; arc CD.

To Prove: $\angle A$ is measured by $\frac{1}{2}$ arc CD.

Proof: I. If one side of the $\angle$ is a diameter.

Draw radius CO. $\triangle AOC$ is isosceles (?).

$\angle COD = \angle A + \angle C$ (102).

$\angle A = \angle C$ (?).

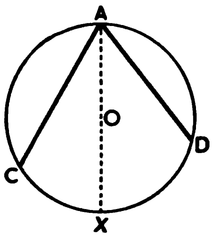
$\therefore \angle COD = \angle A + \angle A = 2\angle A$ (Ax. 6).

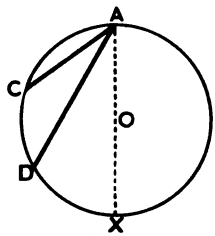
That is, $\frac{1}{2}\angle COD = \angle A$ (Ax. 3).

But $\angle COD$ is meas. by arc CD (232).

$\therefore \frac{1}{2}\angle COD$ is meas. by $\frac{1}{2}$ arc CD (Ax. 3).

$\therefore \angle A$ is meas. by $\frac{1}{2}$ arc CD (Ax. 6).





II. If the center is *within* the angle. Draw diameter AX.

$\angle CAX$ is measured by $\frac{1}{2}$ arc CX (I).

$\angle DAX$ is measured by $\frac{1}{2}$ arc DX (I). Adding,

$\therefore \angle CAD$ is measured by $\frac{1}{2}$ arc CD (Ax. 2).

III. If the center is *without* the angle. Draw diameter AX.

$\angle CAX$ is measured by $\frac{1}{2}$ arc CX (I).

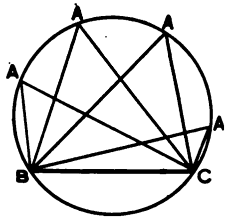
$\angle DAX$ is measured by $\frac{1}{2}$ arc DX (I). Subtracting,

$\therefore \angle CAD$ is measured by $\frac{1}{2}$ arc CD (Ax. 2). Q.E.D.

237. COROLLARY. **Angles measured by half the same arc, or halves of equal arcs, are equal.**

238. COROLLARY. **In the same circle (or in equal circles), equal angles are measured by equal arcs.**

PROPOSITION XXIV. THEOREM



239. **All angles inscribed in the same segment are equal.**

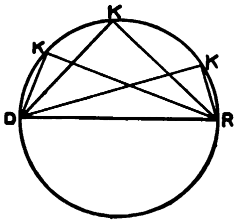
Given: The several $\angle s\ A$ inscribed in segment BAC.

To Prove: These angles all equal.

Proof: Each $\angle\ BAC$ is measured by $\frac{1}{2}$ arc BC (236).

$\therefore$ these angles are all equal. (237).

Q.E.D.



240. COROLLARY. **All angles inscribed in a semicircle are right angles.**

Proof: Each $\angle\ K$ is measured by $\frac{1}{2}$ a semicircle (236).

$\therefore$ each $\angle\ K =$ a rt. $\angle$ (234).

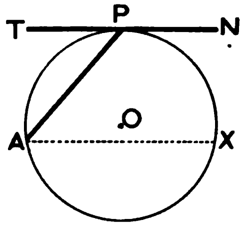
Q.E.D.

Historical Note. Thales, a Greek from Asia Minor, studied geometry from the Egyptians in the sixth century B.C. He discovered the truth of 240 as well as a number of very important theorems. For example: Book I, Propositions I, II, IV and XXXIV, and Book III, Proposition XX.


THALES

Thales was one of the "Seven Wise Men of Greece," and made important contributions to astronomy and philosophy as well as to geometry. He regarded water as the principle of all things.

PROPOSITION XXV. THEOREM



241. The angle formed by a tangent and a chord is measured by half the intercepted arc.

Given: ⊙ O, tangent TN; chord AP; $\angle TPA$; arc PA.

To Prove: $\angle TPA$ is measured by $\frac{1}{2}$ arc PA.

Proof: Through A suppose AX drawn ∥ to TN.

Now $\angle A$ is meas. by $\frac{1}{2}$ arc PX		(236).
But	$\angle A = \angle TPA$	(66).
Also	arc PX = arc PA	(220).
Substituting, $\angle TPA$ is meas. by $\frac{1}{2}$ arc PA		(Ax. 6).

Q.E.D.

Ex. 1. A chord divides a circle into two arcs, one containing 100°, the other, 260°. An angle is inscribed in each segment. How many degrees are there in each angle?

Ex. 2. In a circle, an inscribed angle and a central angle intercept the same arc, which contains 140°. How many degrees are there in each angle?

Ex. 3. A chord subtends an arc of 74°. How many degrees are there in the angle between the chord and a tangent at one extremity?

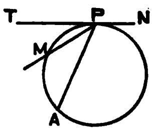
Ex. 4. The circumference of a circle is divided into four arcs, 40°, 70°, 100°, and x. Find x and the angles of the quadrilateral formed by the chords of these arcs.

Ex. 5. In a segment of a circle whose arc contains 210° is inscribed an angle. How many degrees are there in this angle?

Ex. 6. An inscribed angle contains 35°. How many degrees are there in its intercepted arc?

Ex. 7. The line bisecting an inscribed angle bisects also its intercepted arc.

Ex. 8. State and prove the converse of Ex. 7.

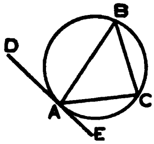


Ex. 9. The line bisecting the angle between a tangent and a chord bisects the intercepted arc.

Ex. 10. State and prove the converse of Ex. 9.

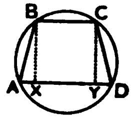
Ex. 11. The angle between a tangent and a chord is half the angle between the radii drawn to the ends of the chord.

Ex. 12. If a triangle is inscribed in a circle and a tangent is drawn at one of the vertices, the angles formed between the tangent and the sides equals the other two angles of the triangle.



Ex. 13. By the figure of Ex. 12 prove that the sum of the three angles of a triangle equals two right angles.

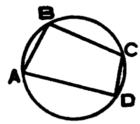
Ex. 14. If one pair of opposite sides of an inscribed quadrilateral are equal, the other pair are parallel.



Proof: Draw $\perp$s BX, CY; arc AB = arc CD (?).
$\therefore$ arc ABC = arc BCD (Ax. 2).
Hence prove rt. $\triangle$s ABX and DCY equal.

Ex. 15. If any pair of diameters is drawn, the lines joining their extremities (in order) form a rectangle.

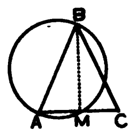
Ex. 16. The opposite angles of an inscribed quadrilateral are supplementary.



Ex. 17. If a tangent and a chord are parallel, and the chords of the two intercepted arcs are drawn, they make equal angles with the tangent.

Ex. 18. The circle described on one of the equal sides of an isosceles triangle as a diameter bisects the base.

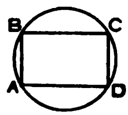
Proof: Draw line BM. The $\triangle$s are rt. $\triangle$s (?) and congruent (?).



Ex. 19. If the circle, described on a side of a triangle as diameter, bisects another side, the triangle is isosceles.

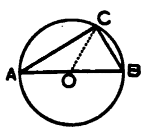
Ex. 20. All angles that are inscribed in a segment greater than a semicircle are acute, and all angles inscribed in a segment less than a semicircle are obtuse.

Ex. 21. An inscribed parallelogram is a rectangle.
Prove arc ABC = arc ADC, etc.

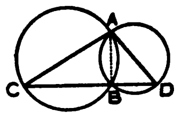


Ex. 22. The diagonal of an inscribed rectangle is a diameter.

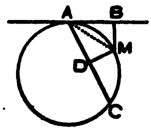
Ex. 23. A circle described on the hypotenuse of a right triangle as a diameter passes through the vertices of all the right triangles having the same hypotenuse.



Ex. 24. If from one end of a diameter a chord is drawn, a perpendicular to it drawn from the other end of the diameter intersects the first chord on the circumference.

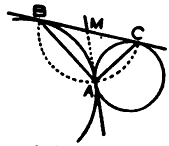


Ex. 25. If two circles intersect and a diameter is drawn in each circle through one of the points of intersection, the line joining the ends of these diameters passes through the other point of intersection. [Draw chord AB.]



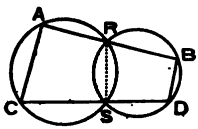
Ex. 26. If a tangent is drawn at one end of a chord, the midpoint of the intercepted arc is equally distant from the chord and the tangent.

[Draw chord AM and prove the rt. ▲ congruent.]

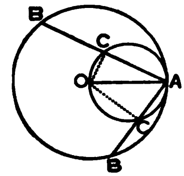


Ex. 27. If two circles are tangent at A and a common tangent touches them at B and C, the angle BAC is a right angle. [Draw tangent at A. Use 206, 240.]

Ex. 28. The bisectors of all the angles inscribed in the same segment of a circle pass through a common point.



Ex. 29. If two circles intersect and a line is drawn through each point of intersection terminating in the circles, the chords joining these extremities are parallel. [Draw RS. $\angle A$ is supp. of $\angle RSC$ (?). Finally use 73.]



Ex. 30. A circle described on the radius of another circle as diameter bisects all chords of the larger circle drawn from their point of contact.

To Prove: AB is bisected at C.

Proof: Draw chord OC. (Use 240, 200).

Ex. 31. If two equal chords intersect within a circle, the segments of one are equal to the segments of the other, each to each.

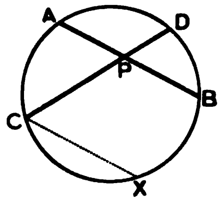
Ex. 32. Prove Proposition XXV by drawing a diameter to the point of tangency, instead of a chord parallel to the tangent.

Ex. 33. If in figure of Ex. 25 above, line CD met the two circles at M and N instead of at a single point B, what could be said of the lines AM and AN?

Ex. 34. If a circle is divided into four equal arcs and if chords of these arcs are drawn, the inscribed figure is a square.

PROPOSITION XXVI. THEOREM

242. The angle formed by two chords intersecting within the circle is measured by half the *sum* of the intercepted arcs. (The arcs are those intercepted by the given angle and by its vertical angle.)



Given: Chords AB and CD intersecting at P; $\angle APC$; arcs AC and DB.

To Prove: $\angle APC$ is measured by $\frac{1}{2}$ (arc AC + arc DB).

Proof: Suppose CX drawn through $C \parallel$ to AB.

Now $\angle C$ is measured by $\frac{1}{2}$ arc DX (236).

That is, $\angle C$ is measured by $\frac{1}{2}$ (arc BX + arc DB).

But $\angle C = \angle APC$ (66).

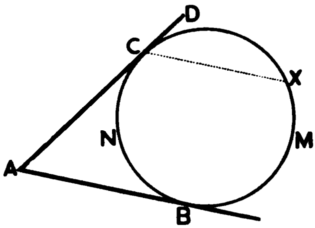
Also arc BX = arc AC (220).

$\therefore \angle APC$ is meas. by $\frac{1}{2}$ (arc AC + arc DB) (Ax. 6).

Q.E.D.

PROPOSITION XXVII. THEOREM

243. The angle formed by two tangents is measured by half the *difference* of the intercepted arcs.



Given: The two tangents AC and AB; $\angle A$; arcs CMB and CNB.

To Prove: $\angle A$ is measured by $\frac{1}{2}$ (arc CMB − arc CNB).

Proof: Suppose CX drawn $\parallel$ to AB.

Now $\angle DCX$ is meas. by $\frac{1}{2}$ arc CX (241).

That is, $\angle DCX$ is meas. by $\frac{1}{2}$ (arc CMB − arc BX).

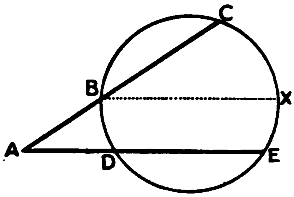
But $\angle DCX = \angle A$ (67).

Also arc BX = arc CNB (220).

$\therefore \angle A$ is meas. by $\frac{1}{2}$ (arc CMB − arc CNB) (Ax. 6).

Q.E.D.

Proposition XXVIII. Theorem



244. The angle formed by two secants which intersect without the circle is measured by half the *difference* of the intercepted arcs.

Given: (?).

To Prove: (?).

Proof: Suppose BX drawn. Where? How?

$\angle CBX$ is meas. by $\frac{1}{2}$ arc CX

That is, $\angle CBX$ is meas. by $\frac{1}{2}$ (arc CE − arc XE) (?).

But $\angle CBX = \angle A$ (67).

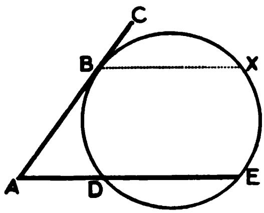
And arc XE = arc BD (?).

Substituting, $\angle A$ is meas. by $\frac{1}{2}$ (arc CE − arc BD)

(Ax. 6).

Q.E.D.

Proposition XXIX. Theorem



245. The angle formed by a tangent and a secant which intersect without the circle is measured by half the *difference* of the intercepted arcs.

Given: (?).

To Prove: (?).

Proof: Suppose BX drawn, etc.

$\angle CBX$ is meas. by $\frac{1}{2}$ arc BX (241).

That is, $\angle CBX$ is meas. by $\frac{1}{2}$ (arc BXE − arc XE).

But $\angle CBX = \angle A$ (?).

And arc XE = arc BD (?).

Substituting, $\angle A$ is meas. by $\frac{1}{2}$ (arc BXE − arc BD) (?).

Q.E.D.

Ex. 1. Where is the vertex of an angle that is measured by one arc? by half an arc? by half the sum of two arcs? by half the difference of two arcs?

Ex. 2. State these truths all in a single theorem of your own.

ORIGINAL EXERCISES

1. The arcs intercepted by two secants intersecting without a circle contain 20° and 140° respectively. How many degrees are there in the angle formed by the secants?

2. If in Ex. 1 the intersecting lines were chords, how many degrees would there be in their angle?

3. One of the arcs intercepted by two intersecting tangents is 72°. Find the angle formed by the tangents.

4. Two intersecting chords intercept opposite arcs of 28° and 80°. How many degrees are there in the angle formed by the chords?

5. The angle between a tangent and a chord contains 27°. How many degrees are there in the intercepted arc?

6. The angle between two chords is 30°; one of the arcs intercepted is 40°. Find the other arc. [Denote the arc by x.]

7. If in figure of 241, arc AP contains 124°, how many degrees are there in $\angle TPA$? in $\angle NPA$? in arc AX?

8. If in figure of 242, arc AC is 85°, $\angle APC$ is 47°, find arc DB.

9. If the arcs intercepted by two tangents contain 80° and 280°, find the angle formed by the tangents.

10. If the arcs intercepted by two secants contain 35° and 185°, find the angle formed by the secants.

11. If in figure of 243, arc CB is 135°, find the angle A.

12. If in figure of 244, angle $A = 42°$ and arc $BD = 70°$, find arc CE.

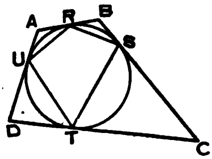
13. If in figure of 245, angle $A = 18°$, arc $BXE = 190°$, find arc BD.

14. If the angle between two tangents is 80°, find the number of degrees in each intercepted arc. [Denote the arcs by x and $360° - x$.]

15. Three of the intercepted arcs of a circumscribed quadrilateral are 68°, 98°, 114°. Find the angles of the quadrilateral. If the chords are drawn connecting (in order) the four points of contact, find the angles of this inscribed quadrilateral. Also find the angles between the diagonals of the two quadrilaterals.

16. If the angle between two tangents to a circle is 40°, find the other angles of the triangle formed by drawing the chord joining the points of contact.

17. The circumference of a circle is divided into four arcs, three of which are, $RS = 62°$, $ST = 142°$, $TU = 98°$. Find:



(1) Arc UR.

(2) The three angles at R; at S; at T; at U.

(3) The angles A, B, C, D of circumscribed quadrilateral.

(4) The angles between the diagonals RT and SU.

(5) The angle between RU and ST at their point of intersection (if produced).

(6) The angle between RS and TU at their intersection.

(7) The angle between AD and BC at their intersection.

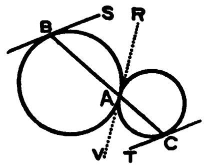
(8) The angle between AB and DC at their intersection.

(9) The angle between RS and DC at their intersection.

(10) The angle between AD and ST at their intersection.

18. If in the figure of Ex. 17, $\angle A = 96°$; $\angle B = 112°$; and $\angle C = 68°$, find the angles of the quadrilateral $RSTU$. [Denote arc RU by x. $\therefore$ in $\triangle ARU$, $96° + \frac{1}{2}x + \frac{1}{2}x = 180°$. $\therefore x =$ etc.]

19. If two circles are tangent externally and any line through their point of contact intersects them at B and C, the tangents at B and C are parallel. [Draw common tangent at A. Prove: $\angle ACT = \angle ABS$.]

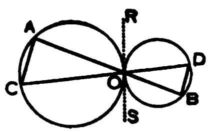


20. Prove the same theorem if the circles are tangent internally.

21. If two circles are tangent externally and any line is drawn through their point of contact terminating in the circles, the two diameters drawn to the extremities are parallel.

22. Prove the same theorem if the circles are tangent internally.

23. If two circles are tangent externally and any two lines are drawn through their point of contact intersecting the circles, the chords joining these points of intersection are parallel.



[Draw common tangent at O. Prove: $\angle C = \angle D$.]

24. Prove the same theorem if the circles are tangent internally.

25. Prove the theorem of 242 by drawing AD instead of CX, and using $\angle APC$ as an exterior angle of $\triangle APD$.

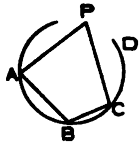
26. Prove the theorem of 243 by drawing BC and using $\angle DCB$ as an exterior angle of $\triangle ABC$.

27. Prove the theorem of 244 by drawing CD and using angle CDE as an exterior angle of triangle ACD.

28. Prove the same theorem by drawing BE.

29. Prove the theorem of 245 by drawing BE.

30. If the opposite angles of a quadrilateral are supplementary, a circle can be drawn circumscribing it.

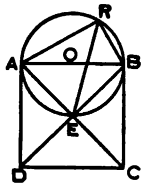


To Prove: A $\odot$ can be drawn through A, B, C, P.

Proof: A $\odot$ can be drawn through A, B, C (?). It is required to prove that it will contain point P. $\angle P + \angle B$ are supp. (?) $\therefore$ they must be meas. by half the entire circle. $\angle B$ is meas. by $\frac{1}{2}$ arc ADC (?). Hence $\angle P$ is meas. by $\frac{1}{2}$ arc ABC. If $\angle P$ is *within* or *without* the circle, it is not meas. by $\frac{1}{2}$ arc ABC. (Why not?)

31. The circle described on the side of a square, or of a rhombus, as a diameter passes through the point of intersection of the diagonals. [Use 135, 141.]

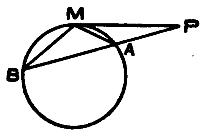
32. The line joining the vertex of the right angle of a right triangle to the point of intersection of the diagonals of the square constructed upon the hypotenuse as a side, bisects the right angle of the triangle.



Proof: Describe a $\odot$ upon the hypotenuse as diameter and use 141, 196, 237.

33. If two secants, PAB and PCD, meet a circle at A, B, C, and D, respectively, the triangles PBC and PAD are mutually equiangular.

34. If PAB is a secant and PM is a tangent to a circle from P, the triangles PAM and PBM are mutually equiangular.



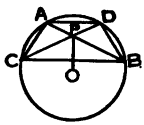
35. If two equal chords intersect within a circle,

(1) One pair of intercepted arcs are equal.

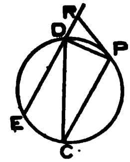
(2) Corresponding parts of the chords are equal.

(3) The lines joining their extremities (in order) form an isosceles trapezoid.

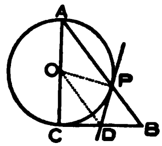
(4) The radius drawn to their intersection bisects their angle.



36. If a secant intersects a circle at D and E, PC is a parallel chord, and PR a tangent at P meeting the secant at R, the triangles PCD and PRD are mutually equiangular. [$\angle R$ and $\angle CDP$ are measured by $\frac{1}{2}$ arc PC. (Explain.) Etc.]

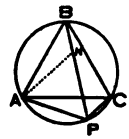


37. If a circle is described upon one leg of a right triangle as diameter and a tangent is drawn at the point of its intersection with the hypotenuse, this tangent bisects the other leg.

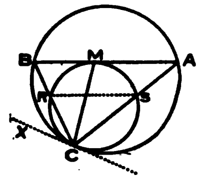


[Draw OP and OD. CD is tangent (?). OD bisects arc PC (207). $\angle COD = \angle A$ (237). $\therefore$ OD is $\parallel$ to AB (?). Etc.]

38. If an equilateral triangle ABC is inscribed in a circle and P is any point of arc AC, $AP + PC = BP$. [Take $PN = PA$; draw AN. $\triangle ANP$ is equilateral. (Explain.) $\triangle ANB = \triangle APC$ (?). Etc.]



39. If two circles are tangent internally at C, and a chord AB of the larger circle is tangent to the less circle at M, the line CM bisects the angle ACB.

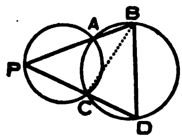


[Draw tangent CX and chord RS. (Explain.)

$$\angle RSC = \angle BCX = \angle A.$$

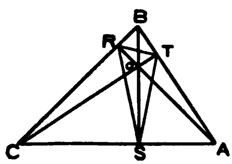
$\therefore$ AB is $\parallel$ to RS (?). Etc.]

40. If two circles intersect at A and C and lines are drawn from any point P, in one circle, through A and C terminating in the other at points B and D, chord BD will be of constant length for all positions of point P.



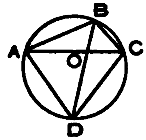
[Draw BC. Prove $\angle BCD$, the ext. $\angle$ of $\triangle PBC$, = a constant. Etc.]

41. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of these perpendiculars.



To Prove: BS bisects $\angle RST$, etc.

Proof: If a circle is described on AO as diam., it will pass through T and S (141). If a circle is described on OC as diam., it will pass through R and S (?). $\therefore$ $\angle BAR = BST$ (?); and $\angle BCT = \angle BSR$ (?). But $\angle BAR = \angle BCT$. (Each is the comp. of $\angle ABC$.) $\therefore$ Etc.



42. If ABC is a triangle inscribed in a circle, BD is the bisector of angle ABC, meeting AC at O and the circle at D, the triangles AOB and COD are mutually equiangular. Also triangles BOC and AOD. Also triangles BOC and ABD. Also triangles AOD and ABD. Also triangles BCD and COD.

43. If two circles intersect at A and B, and from P, any point on one of them, lines AP and BP are drawn cutting the other circle again at C and D respectively, CD is parallel to the tangent at P.

44. If two circles intersect at A and B, and through B a line is drawn meeting the circles at R and S respectively, the angle RAS is constant for all positions of the line RS.

[Prove $\angle R + \angle S$ is constant. $\therefore \angle RAS$ is also constant.]

45. Two circles intersect at A, and through A any secant is drawn meeting the circles again at M and N. Prove that the tangents at M and N meet at an angle which remains constant for all positions of the secant.

[Prove the angle between these tangents equal to the angle between the tangents to the circles at A.]

46. Two equal circles intersect at A and B, and through A any straight line MAN is drawn, meeting the circles at M and N respectively. Prove chord BM = chord BN.

47. If the midpoint of the arc subtended by any chord is joined to the extremities of any other chord,

(1) The triangles formed are mutually equiangular. (2) The opposite angles of the quadrilateral thus formed are supplementary.

48. Two circles meet at A and B and a tangent to each circle is drawn at A, meeting the circles at R and S respectively. Prove that the triangles ABR and ABS are mutually equiangular.

49. Two chords intersecting within a circle divide the circumference into parts that bear the relation to each other of 1, 2, 3, 4. Find the angles made by the chords. [Denote the arcs by x, $2x$, $3x$, $4x$.]

50. If $ABCD$ is an inscribed quadrilateral, AB and DC produced to meet at E, AD and BC produced to meet at F, the bisectors of angles E and F are perpendicular.

[The difference of one pair of arcs = difference of a second pair; the difference of a third pair = difference of a fourth pair. (Explain.) Transpose negative terms and add correctly, noting that the sum of 4 arcs = sum of 4 others, and hence = 180°. Half the sum of these 4 arcs measures the angle between the bisectors. (Explain.) Etc.]

LOCI

246. The **locus** of a point is the series of positions the point must occupy in order that it may satisfy a given condition. It is the path of a point whose positions are limited or defined by a given condition, or given conditions.

247. Explanatory. I. If a point is moving so that it is always one inch from a given point, the moving point may occupy any position in a circle whose center is the fixed point and whose radius is one inch. Furthermore, this moving point cannot occupy any position outside of the circle, or its position will not fulfill the given condition.

THEOREM. The locus of points in a plane a given distance from a given point is a circle the center of which is the given point and the radius of which is the given distance.

II. If a point is moving so that it is always equally distant from the ends of a straight line, it must move in the perpendicular bisector of the line.

THEOREM. **The locus of points equally distant from the ends of a line is the perpendicular bisector of the line.**

Proof: Every point *in* the $\perp$ bisector is equally distant from the ends of the line. (80.)

No point *without* this $\perp$ fulfills that condition. (81.)

$\therefore$ the $\perp$ bisector is the locus. (246.)

III. THEOREM. The locus of points equally distant from the sides of an angle is the bisector of the angle.

Proof: Any point within the bisector of an angle is equally distant from the sides. (94.)

Any point equally distant from the sides of an angle lies in the bisector. (95.)

Hence all the points in the bisector fulfill the condition and there are no other points that fulfill it.

That is, the bisector is the locus, etc. Q.E.D.

IV. The method of proving that a certain line or a group of lines is the locus of points satisfying a given condition, consists in proving that *every point* in the line fulfills the given requirement, and that there is *no other point* that fulfills it. In the above illustrations it is evident that every point in the lines that were called the "locus," fulfilled the conditions of the case. It is evident also that there is no point outside these "loci" that does so fulfill the conditions. That is, these "loci" contain *all* the points described.

ORIGINAL EXERCISES ON LOCI

1. What is the locus of a point so moving that it is always two feet away from a given line?

2. What is the locus of a point so moving that it is always equally distant from two parallel lines?

3. What is the locus of points equally distant from two given fixed points?

4. If all the radii of a circle were drawn, what would be the locus of their midpoints?

5. If all possible lines were drawn from a vertex of a triangle and terminating in the opposite side, what would be the locus of their midpoints?

6. What is the locus of the midpoints of a series of parallel chords in a circle? Prove.

7. What is the locus of the midpoints of all chords of the same length in a given circle? Prove.

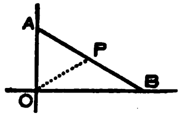
8. What is the locus of all points from which two equal tangents can be drawn to two circles which are tangent to each other?

9. What is the locus of all points at a given distance from a given circumference? Discuss if the distance is > radius. If it is less.

10. What is the locus of the vertices of the right angles of all the right triangles that can be constructed on a given hypotenuse? Prove.

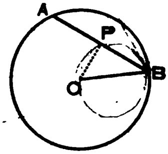
11. What is the locus of the vertices of all the triangles which have a given acute angle (at that vertex) and have a given base? Prove.

12. A line of given length moves so that its ends are in two perpendicular lines. What is the locus of its midpoint? Prove.



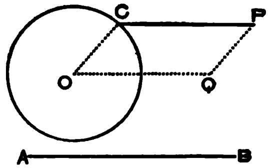
[Suppose AB represents one of the positions of the moving line. Draw OP to its midpoint. In all the positions of AB, $OP = \frac{1}{2} AB =$ a constant (141). $\therefore P$ is always a fixed distance from O. Etc.]

13. What is the locus of the midpoints of all the chords that can be drawn through a fixed point on a given circle?



[Suppose AB represents one of the chords from B in circle O, with radius OB; and P is the midpoint of AB. Draw OP. $\angle P$ is a rt. $\angle$ (?). That is, wherever the chord may be drawn, $\angle P$ is a rt. $\angle$.
$\therefore$ locus of P is, etc.]

14. A definite line which is always parallel to a given line moves so that one of its extremities is on a given circle. Find the locus of the other extremity.



[Suppose CP represents one position of the moving line CP. Draw $OQ =$ and $\parallel$ to CP from center O. Join OC and PQ. Wherever CP is, this figure is a ▱ (?). Its sides are of constant length (?). That is, P is always a fixed distance from Q, etc.]

15. What is the locus of the centers of all circles tangent to a given line at a given point? to a given circle at a given point?

16. A parallelogram, $ABCD$, is hinged at the vertices, and AB only is fixed in position. What is the locus of vertex C? of vertex D? of the midpoint of BC? of the midpoint of CD?

248. Heretofore only a few of the simplest exercises in construction have been given (pages 8–12), and formal proofs of these were not required.

The following methods for constructing lines are given so that mathematical precision may be employed in drawing accurate diagrams of a complex nature. No construction is considered valid unless a proof of its correctness can be given.

The pupil should be familiar with the use of the ruler and compasses.

CONSTRUCTION PROBLEMS

249. A geometrical **construction** is a diagram made of points and lines.

250. A geometrical **problem** is the statement of a required construction. Thus, "It is required to bisect a line" is a problem. A problem is sometimes defined as "a question to be solved" and includes other varieties besides those involved in geometry.

251. The word **proposition** is used to include both *theorem* and *problem*.

252. The complete solution of a problem consists of *five* parts:

I. The **Given** data are to be described.

II. The **Required** construction is to be stated.

III. The **Construction** is to be outlined.

This part usually contains the verb only in the *imperative*. The only limitation in this part of the process is that every construction demanded shall have been shown possible by previous constructions or postulates. (See 32, 33, 186.)

IV. The **Statement** that the required construction has been completed.

V. The **Proof** of this declaration.

Frequently a **discussion** of ambiguous or impossible instances is necessary.

253. Notes. (1) A straight line is determined by two points.

(2) A circle is determined by three points.

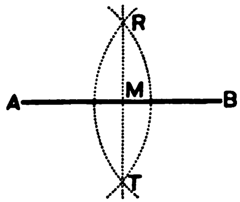
(3) A circle is determined by its center and its radius. Whenever a circumference, or even an arc, is to be drawn, it is *essential* that the **center** and the **radius** be mentioned.

(4) "Q.E.F." = *Quod erat faciendum* — "which was to be done." These letters follow the statement that the construction which was required has been accomplished.

Proposition XXX. Problem

254. To bisect a given line.

Given: The definite line *AB*.



Required: To bisect *AB*.

Construction: Using *A* and *B* as centers and *one* radius, sufficiently long to make the circumferences intersect, describe two arcs meeting at *R* and *T*. Draw *RT* meeting *AB* at *M*.

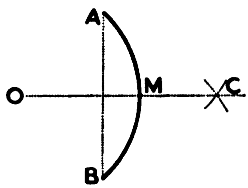
Statement: Point *M* bisects *AB*. Q.E.F.

Proof:	*R* is equally distant from *A* and *B*	(188).
	T is equally distant from *A* and *B*	(?).
Hence	*RT* is the $\perp$ bisector of *AB*	(83).
That is,	*M* bisects *AB*.	Q.E.D.

Proposition XXXI. Problem

255. Problem. To bisect a given arc.

Given: Arc *AB* whose center is *O*.



Required: To bisect arc *AB*.

Construction: Draw chord *AB*. Using *A* and *B* as centers and any sufficient radius, describe arcs meeting at *C*. Draw *OC* cutting arc *AB* at *M*.

Statement: The point *M* bisects arc *AB*. Q.E.F.

Proof: *O* and *C* are each equally distant from *A* and *B* (188).

$\therefore$ *OC* is the $\perp$ bisector of chord *AB* (83).

$\therefore$ *M* bisects arc *AB* (200).

Q.E.D.

Ex. 1. Construct the supplement of a given angle.

Ex. 2. Divide a given line into four equal parts.

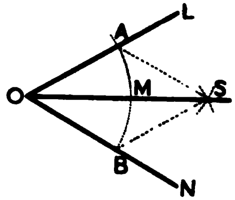
Ex. 3. Divide a given arc into four equal arcs.

PROPOSITION XXXII. PROBLEM

256. To bisect a given angle.

Given: $\angle LON$.

Required: To bisect $\angle LON$.



Construction: Using O as a center and any radius, draw arc AB, cutting LO at A and NO at B. Using A and B as centers and any sufficient radius, draw two arcs intersecting at S. Draw OS meeting arc AB at M.

Statement: OS bisects $\angle LON$. Q.E.F.

Proof: Draw AS and BS. In $\triangle$ AOS and BOS, $OS = OS$ (?).

$OA = OB$ and $AS = BS$ (188).

$\therefore \triangle AOS \cong \triangle BOS$ (?).

$\therefore \angle AOS = \angle BOS$ (?).

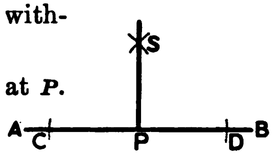
Q.E.D.

PROPOSITION XXXIII. PROBLEM

257. At a fixed point in a straight line to erect a perpendicular to that line.

Given: Line AB and point P within it.

Required: To erect a line $\perp$ to AB at P.



Construction: Using P as a center and any radius, draw arcs meeting AB at C and D. Using C and D as centers and a radius longer than before, draw arcs meeting at S. Draw PS.

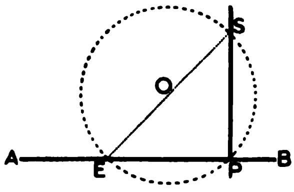
Statement: PS is $\perp$ to AB at P. Q.E.F.

Proof: Points P and S are each equally distant from C and D. (188).

$\therefore PS$ is the $\perp$ bisector of CD (83).

That is, PS is $\perp$ to AB. Q.E.D.

Another Construction: Using any point *O*, without *AB*, as center, and *OP* as radius, describe a circumference, cutting *AB* at *P* and *E*. Draw diameter *EOS*. Join *SP*.



Statement: *SP* is ⊥ to *AB* at *P*. Q.E.F.

Proof: Segment *SPE* is a semicircle (191).

∴ ∠ *SPE* is a rt. ∠ (240).

∴ *SP* is ⊥ to *AB* (16). Q.E.D.

Ex. 1. Construct a right angle.

Ex. 2. Construct an angle of 45°; of 135°; of 22° 30′; of 67° 30′.

Ex. 3. Construct the complement of a given angle.

Ex. 4. Find the center of a given circle.

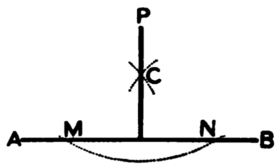
Ex. 5. Construct the second acute angle of a rt. △ if one is known.

Ex. 6. Construct a chord of a circle if its midpoint is known.

Proposition XXXIV. Problem

258. Through a point without a line to draw a perpendicular to that line.

Given: Line *AB* and point *P* without it.



Required: (?).

Construction: Using *P* as a center and any sufficient radius, describe an arc intersecting *AB* at *M* and *N*. Using *M* and *N* as centers and any sufficient radius, describe arcs intersecting each other at *C*. Draw *PC*.

Statement: *PC* is ⊥ to *AB* from *P*. Q.E.F.

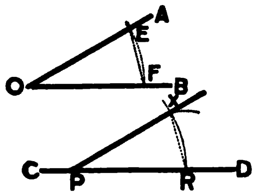
Proof: *P* and *C* are each equally distant from *M* and *N* (188).

∴ *PC* is the ⊥ bisector of *MN* (83).

That is, *PC* is ⊥ to *AB* from *P*. Q.E.D.

PROPOSITION XXXV. PROBLEM

259. At a given point in a given line to construct an angle which shall be equal to a given angle.



Given: $\angle AOB$; point P in line CD.

Required: To construct at P an $\angle =$ to $\angle AOB$.

Construction: Using O as a center with any radius, describe an arc cutting OA at E and OB at F. Draw chord EF. Using P as a center and OE as a radius, describe an arc cutting CD at R. Using R as a center and chord EF as a radius, describe an arc cutting the former arc at X. Draw PX and chord RX.

Statement: $\angle XPD = \angle AOB$. Q.E.F.

Proof: $OE = PX$ and $OF = PR$ and $EF = XR$ (188).

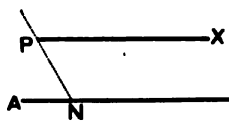
$\therefore \triangle OEF \cong \triangle PXR$ (?).

$\therefore \angle XPD = \angle O$ (?).

Q.E.D.

PROPOSITION XXXVI. PROBLEM

260. To draw a line through a given point parallel to a given line.



Given: Point P and line AB.

Required: To draw through P, a line $\parallel$ to AB.

Construction: Draw through P any line PN, meeting AB at N.

On this line, at P, construct $\angle NPX =$ to $\angle ANP$ (259).

Statement: PX is $\parallel$ to AB. Q.E.F.

Proof: $\angle NPX = \angle ANP$ (Const.).

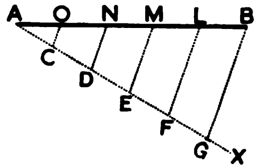
$\therefore PX$ is $\parallel$ to AB (70).

Q.E.D.

Proposition XXXVII. Problem

261. To divide a line into any number of equal parts.

Given: Definite line AB.



Required: To divide it into five equal parts.

Construction: Draw through A any other line AX. On this take any length AC as a unit, and mark off on AX five of these units, AC, CD, DE, EF, FG. Draw GB.

Through F, E, D, C, draw ∥ to GB, lines FL, EM, DN, CO.

Statement: Then $AO = ON = NM = ML = LB$. Q.E.F.

Proof: $AC = CD = DE = EF = FG$ (Const.).

$\therefore AO = ON = NM = ML = LB$ (140).

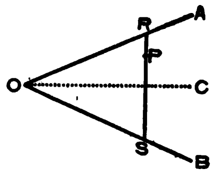
Q.E.D.

Ex. 1. Find a point in one side of a triangle equally distant from the other sides.

Ex. 2. Construct the shortest chord that can be drawn through a given point within a circle.

Proof: Draw any other chord through the point, etc.

Ex. 3. Draw through a given point without a given line, another line which shall make a given angle with the line.



Construction: Construct an ∠ at any point in the given line = to given ∠, etc.

Ex. 4. Construct through a given point a line which makes equal angles with two intersecting lines.

Ex. 5. Bisect the three sides of a triangle, and draw the medians. Do they meet in a point?

Ex. 6. Draw accurately the three altitudes of a triangle. Do they meet in a point?

Ex. 7. Draw accurately the three perpendicular bisectors of the sides of a triangle. Do they meet in a point?

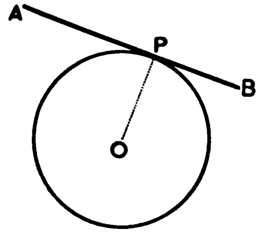
Ex. 8. Draw accurately the three bisectors of the angles of a triangle. Do they meet in a point?

PROPOSITION XXXVIII. PROBLEM

262. To draw a tangent to a given circle through a given point:

I. If the point is on the circle.

II. If the point is without the circle.



I. **Given:** ⊙ O; P, a point on the circle.

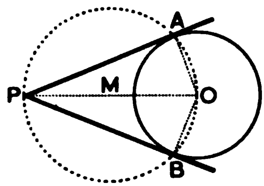
Required: To draw a tangent through P.

Construction: Draw the radius OP. Draw line $AB \perp$ to OP at P (by 257).

Statement: AB is tangent to ⊙ O at P. Q.E.F.

Proof: AB is $\perp$ to OP at P (Const.).

$\therefore AB$ is a tangent (202). Q.E.D.



II. **Given:** ⊙ O; P, a point without it.

Required: To draw a tangent through P.

Construction: Draw PO; bisect it at M (by 254).

Using M as a center and PM as a radius, describe a circumference intersecting ⊙ O at A and B.

Draw PA, PB, OA, OB.

Statement: PA and PB are tangents through P Q.E.F.

Proof: ⊙ M passes through O ($PM = MO$ by const.).

$\therefore < PAO$ is a rt. $\angle$ (240).

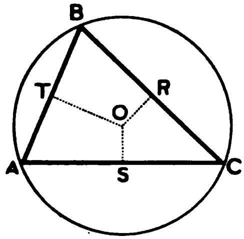
$\therefore PA$ is a tangent (202).

Similarly, PB is a tangent. Q.E.D.

NOTE. The student has probably observed that in constructions certain lines and angles *must precede* others. The order of the successive steps is an important consideration. Thus, in 261 and 262 certain lines must be drawn before others *can* be drawn.

PROPOSITION XXXIX. PROBLEM

263. To circumscribe a circle about a given triangle.



Given: (?).

Required: (?). (See 214.)

Construction: Bisect *AB*, *BC*, *AC*. Erect ⊥s at *T*, *R*, *S*, meeting at *O*.

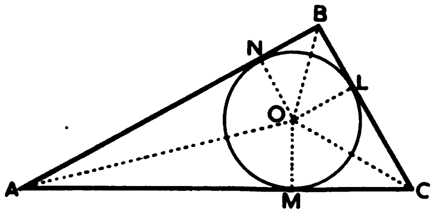
Using *O* as a center and *OA* as radius, draw a circle.

Statement: This ⊙ will pass through vertices *A*, *B*, and *C*. Q.E.F.

Proof: (100). Q.E.D.

PROPOSITION XL. PROBLEM

264. To inscribe a circle in a given triangle.



Given: (?). **Required:** (?).

Construction: Draw the three bisectors of the ∠s of △ *ABC*, meeting at *O* (by 256). Draw ⊥s from *O* to the three sides.

Using *O* as a center and one ⊥ as a radius, draw a ⊙.

Statement: This ⊙ will be tangent to the three sides of △ *ABC*. Q.E.F.

Proof: The bisectors of these angles meet in a point and the ⊥s *OL*, *OM*, *ON* are equal (99).

∴ the circumference passes through *L*, *M*, *N* (179).

Also the three sides are tangent to the ⊙ (202).

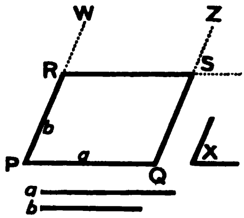
That is, the ⊙ *O* is inscribed in △ *ABC* (221).

Q.E.D.

265. If a circle is described tangent to one side of a triangle and tangent to the prolongations of the other sides, it is called an **escribed circle.** Every triangle may have three escribed circles.

PROPOSITION XLI. PROBLEM

266. To construct a parallelogram if two sides and the included angle are given.



Given: The sides a and b and their included angle, x.

Required: To construct a ▱ containing these parts.

Construction: Take a straight line $PQ =$ to a.

At P construct $\angle P =$ to $\angle x$. On PW, the side of this $\angle$, take $PR =$ to b.

At R draw $RY \parallel$ to PQ; and at Q draw $QZ \parallel$ to PW.

Denote the intersection of these lines by S.

Statement: $PQSR$ is the required parallelogram. Q.E.F.

Proof: First, it is *a* parallelogram. (Def.).

Second, it is *the required* parallelogram. (Because it contains the given parts.) Q.E.D.

Ex. 1. Draw a triangle and all its exterior angles. Bisect these to find centers of escribed circles. What are the radii of these circles? Draw the three escribed circles.

Ex. 2. Draw a rectangle, having given two sides.

Ex. 3. What several things must be known about a circle before you can draw a tangent?

Ex. 4. Is a radius of a circle a tangent? Why?

Ex. 5. How can one find the center of a given arc? Is this the same as its midpoint?

Ex. 6. Give another method of solving Proposition XXXVII.

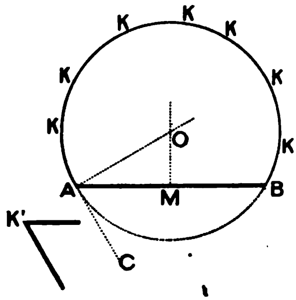
Ex. 7. If a hundred triangles stood on the same base, and all their vertex angles were equal, where would all their vertices be?

PROPOSITION XLII. PROBLEM

267. To construct a segment of a circle upon a given line, as chord, which shall contain angles equal to a given angle.

Given: Line AB and $\angle K'$.

Required: To construct a segment upon AB the inscribed angles of which shall $= \angle K'$.



Construction: Construct at A, $\angle BAC =$ to $\angle K'$.

Bisect AB at M.

At M erect $OM \perp$ to AB.

At A erect $OA \perp$ to AC, meeting OM at O.

Using O as a center and OA as radius, describe $\odot\ O$.

Statement: The $\angle$s inscribed in segment $AKB = \angle K'$. Q.E.F.

Proof: The circle passes through B (80).

$\therefore AB$ is a chord (181).

AC is a tangent to the $\odot$ (202).

$\therefore \angle BAC$ is meas. by half the arc AB (241).

Any angle inscribed in AKB is meas. by half arc AB (236).

$\therefore$ any angle $AKB = \angle BAC$ (237).

$\therefore$ any inscribed $\angle AKB = \angle K'$ (Ax. 1).

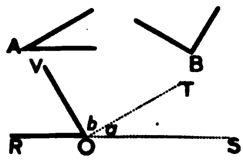
Q.E.D.

[Let the pupil draw chords AK and BK, which were purposely omitted.]

Historical Note. Substantial contributions were made to the advancement of geometrical science by Hippocrates, a Greek philosopher, who, during the fifth century B.C., discovered many properties of the circle. He was the first to employ the method of proof known as the *reductio ad absurdum*, and he wrote the first text book on geometry. He was not aware of the truth that equal central angles and equal inscribed angles intercept equal arcs, although he knew that areas of circles are proportional to the squares of their radii.

PROPOSITION XLIII. PROBLEM

268. To construct the third angle of a triangle if two angles are known.



Given: $\angle$s A and B, two $\angle$s of a $\triangle$.

Required: To construct the third.

Construction: At point O in line RS construct $\angle a =$ to $\angle A$.

At point O in OT construct $\angle b =$ to $\angle B$.

Statement: The $\angle VOR =$ the third $\angle$ of the $\triangle$. Q.E.F.

Proof: $\angle a + \angle b + \angle VOR = 2$ rt. $\angle$s (46).

$\angle A + \angle B +$ the third $\angle$ of the $\triangle = 2$ rt. $\angle$s (104).

$\therefore \angle a + \angle b + \angle VOR = \angle A + \angle B +$ the third $\angle$ (Ax. 1).

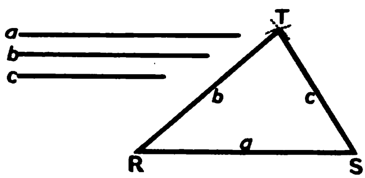
But $\angle a + \angle b = \angle A + \angle B$ (Const. and Ax. 2).

Subtracting, $\angle VOR =$ the third $\angle$ of the $\triangle$ (Ax. 2).

Q.E.D.

PROPOSITION XLIV. PROBLEM

269. To construct a triangle if the three sides are known.



Given: Sides a, b, c of a $\triangle$.

Required: To construct the $\triangle$.

Construction:

Draw $RS =$ to a.

Using R as a center and b as a radius, describe an arc. Using S as a center and c as a radius, describe an arc intersecting the former arc at T. Draw RT and ST.

Statement: $\triangle RST$ is the required $\triangle$. Q.E.F.

Proof: RST is a $\triangle$ (23).

RST is *the required* $\triangle$. (It contains a, b, c.) Q.E.D.

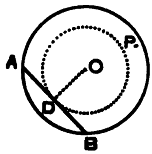
Discussion: Is this problem ever impossible? When?

Ex. 1. Construct an angle of 60°; of 30°; of 15°; of 7° 30′; of 75°.

Ex. 2. Trisect a right angle.

Ex. 3. Construct a tangent to a circle, parallel to a given line.

Ex. 4. Construct a tangent to a given circle perpendicular to a given line.

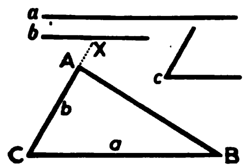


Ex. 5. Construct through a given point within a circle, two chords each equal to a given chord. Is this ever impossible?

Ex. 6. Construct in a given circle a chord equal to a second chord and parallel to a third.

Proposition XLV. Problem

270. To construct a triangle if two sides and the included angle are known.



Given: The sides a and b, and their included $\angle c$ in a $\triangle$.

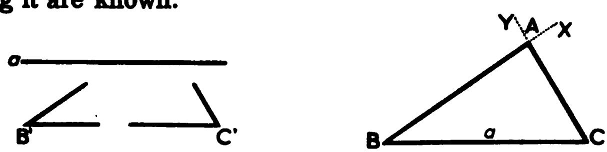
Required: To construct the $\triangle$.

Construction: Draw $CB =$ to a. At C construct $\angle BCX =$ to given $\angle c$. On CX take $CA =$ to b. Join AB.

Statement: (?). **Proof:** (?). **Discussion:** (?).

Proposition XLVI. Problem

271. To construct a triangle if a side and the two angles adjoining it are known.



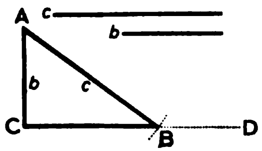
Given: (?). **Required:** (?).

Construction: Draw $BC =$ to a. At B construct $\angle CBX =$ to $\angle B'$; at C construct $\angle BCY =$ to $\angle C'$. Denote the point of intersection of BX and CY by A.

Statement: (?). **Proof:** (?). **Discussion:** (?).

PROPOSITION XLVII. PROBLEM

272. To construct a right triangle if the hypotenuse and a leg are known.



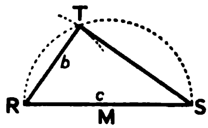
Given: Hypotenuse c; leg b.

Required: (?).

Construction: Draw an indefinite line CD and at C erect a $\perp =$ to b. Using A as a center and c as a radius, describe an arc cutting CD at B. Draw AB.

Statement: (?). **Proof:** (?). **Discussion:** (?).

Another Construction: Take a line, $RS =$ to c. With its midpoint M, as center, and RM as radius describe a semicircle. With R as center and b as radius, describe an arc cutting the semicircumference at T. Draw TR and TS.



Statement: (?). **Proof:** (?). **Discussion:** (?).

PROPOSITION XLVIII. PROBLEM

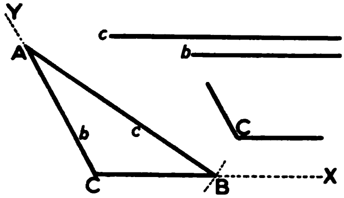
273. To construct a triangle if an angle, a side adjoining it, and the side opposite it are known; that is, if two sides and an angle opposite one of them are known.

The known angle may be obtuse, right, or acute. Consider:
First, If "side opposite" > "side adjoining."
Second, If "side opposite" = "side adjoining."
Third, If "side opposite" < "side adjoining."

Construction for all: Draw an indefinite line, CX, and at one extremity construct an $\angle =$ to $\angle C$. Take on the side of this angle a distance from the vertex equal to the "side adjoining." Using the end of this side as a center and the "side opposite" as a radius, describe an arc intersecting CX. Draw radius to the intersection just found.

If the known angle is obtuse or right.

Given: $\angle C$, side c opposite it, and side b adjoining $\angle C$.



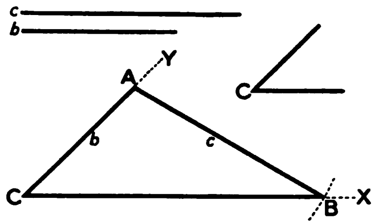
Construction: As above.

Discussion: *Case I.* $c > b$.
The $\triangle$ is always possible.

Case II. $c = b$.
The $\triangle$ is never possible (106).

Case III. $c < b$.
The $\triangle$ is never possible (116).

If the known angle is acute.

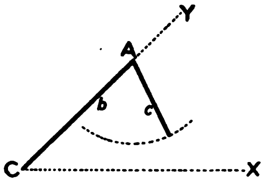


Case I. $c > b$.
The $\triangle$ is always possible.

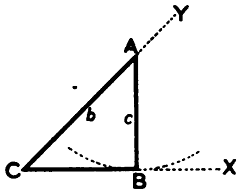
Case II. $c = b$.
The $\triangle$ is always possible and isosceles.

Case III. $c < b$.

First, $c <$ the $\perp$ from A to CX.
The $\triangle$ is never possible.

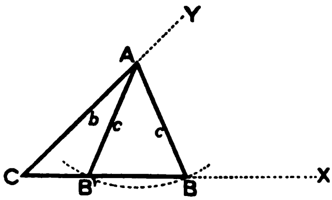


Second, $c =$ the $\perp$ from A to CX.
The $\triangle$ is possible and a right $\triangle$.



Third, $c >$ the $\perp$ from A to CX.

There are *two* $\triangle$s, ACB and ACB'; both contain the three given parts.



ANALYSIS

Many constructions are so simple that their correct solution will readily occur to the pupil.

Sometimes, in the case of complicated constructions, the pupil must have the ability to put the given parts together, one by one. The following outline may be found helpful if employed intelligently. It is called the method of **analysis.**

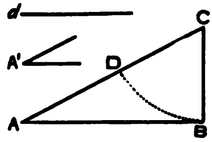
I. Suppose the construction made, that is, suppose the figure drawn.

II. Study this figure in search of truths by which the **order** of the lines that have been drawn can be determined.

III. One or more auxiliary lines may be necessary.

IV. Finally, construct the figure and prove it correct.

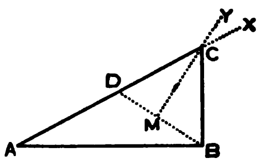
Exercise. Given the base of a triangle, an adjacent acute angle, and the difference of the other sides, to construct the triangle.



Given: Base AB; $\angle A'$; difference d.

Required: To construct the $\triangle$.

[**Analysis:** Suppose $\triangle\ ABC$ is the required $\triangle$. It is evident if $CD = CB$, they may be sides of an isos. $\triangle$ and $AD = d$.]



Construction: At A on AB construct $\angle BAX =$ to $\angle A'$ and on AX, take $AD =$ to d. Join DB. At M, midpoint of DB, draw $MY \perp$ to DB meeting AX at C. Draw CB.

Statement: (?). **Proof:** (?). **Discussion:** (?).

Historical Note. The philosopher Plato and his school flourished at Athens in the fourth century B.C. Plato brought to geometry exact definitions and axioms as well as the method of *analysis*, which is helpful in discovering difficult proofs and constructions.

ORIGINAL CONSTRUCTIONS

I. Construct an **isosceles triangle**, having given:

1. The base and one of the equal sides.

2. The base and one of the equal angles.

3. One of the equal sides and the vertex angle.

4. One of the equal sides and one of the equal angles.

5. The base and the altitude upon it.

6. The base and the radius of the inscribed circle.

[Bisect the base; erect a $\perp$ = to the radius; describe $\odot$, etc.]

7. The base and the radius of the circumscribed circle.

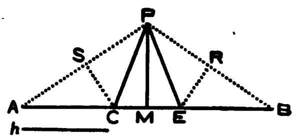
8. The altitude and the vertex angle.

9. The base and the vertex angle.

[Find the supplement of the given $\angle$; bisect this; at each end of base construct an $\angle$ = to this half; etc.]

10. The perimeter and the altitude.

Given: Perimeter $= AB$; alt. $= h$. **Required**: (?). **Construction**: Bisect AB; erect at M $\perp = $ to h; draw AP and BP. Bisect these; erect $\perp$s SC and RE; etc.



II. Construct a **right triangle**, having given:

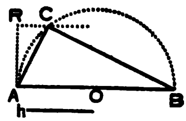
11. The two legs.

12. One leg and the adjoining acute angle.

13. One leg and the opposite acute angle.

14. The hypotenuse and an acute angle.

15. The hypotenuse and the altitude upon it.

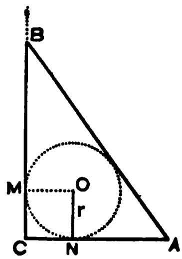


16. The median and the altitude upon the hypotenuse.

17. The radius of the circumscribed circle and a leg.

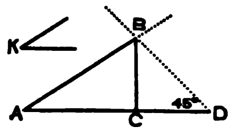
18. The radius of the inscribed circle and a leg.

Given: Radius $= r$; leg $= CA$. **Required**: (?). **Analysis**: Consider that ABC is the completed figure; $CNOM$ is a square, whose vertex O is the center of the circle, and side ON is the given radius. AB is tangent from A. **Construction**: On CA take $CN =$ to r and construct square, $CNOM$. Prolong CM indefinitely. Describe $\odot$, etc.



19. One leg and the altitude upon the hypotenuse.

20. An acute angle and the sum of the legs.

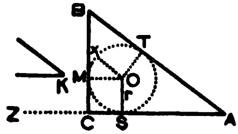


Given: $AD =$ sum; $\angle K$. **Required**: (?).
Construction: At A construct $\angle A =$ to $\angle K$; at D construct $\angle D =$ to 45°, etc.

21. The hypotenuse and the sum of the legs.
[Use A as center, hypotenuse as radius, etc.]

22. The radius of the circumscribed circle and an acute angle.

23. The radius of the inscribed circle and an acute angle.



Construction: Take $CS =$ to r, on indefinite line, ZA. On CS construct square $CSOM$. At O construct $\angle MOX =$ to $\angle K$. Draw radius OT $\perp$ to OX. Draw tangent at T, etc.

III. Construct an **equilateral triangle**, having given:

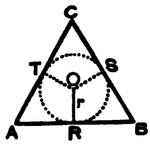
24. One side.

25. The altitude.

26. The perimeter.

27. A median.

28. The radius of the inscribed circle.



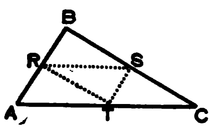
[Draw circle and radius; at center construct $\angle ROS$ $=$ to 120° and $\angle ROT =$ to 120°; etc.]

29. The radius of the circumscribed circle.

IV. Construct a **triangle**, having given:

30. The base, an angle adjoining it, the altitude upon it.

31. The midpoints of the three sides.
[Draw RS, RT, ST, etc.]



32. One side, altitude upon it, and the radius of the circumscribed circle.

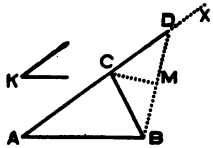
Construction: Draw ⊙ with given radius and any center. Take chord $=$ to given side; etc.

33. One side, an adjoining angle, and the radius of the circumscribed circle.

34. Two sides and the altitude from the same vertex.

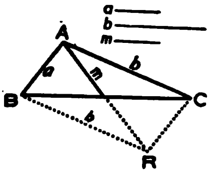
Construction: Erect $\perp$ equal to altitude, upon an indefinite line. Use the end of this altitude as center, and the given sides as radii, etc.

35. One side, an angle adjoining it, and the sum of the other two sides.



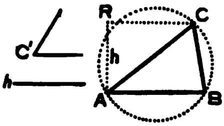
Construction: At A construct $\angle BAX =$ to given $\angle K$. On AX take $AD =$ to s; draw DB; bisect DB at M, etc.

36. Two sides and the median to the third side.



Given: a, b, m. **Construction**: Construct $\triangle ABR$ with three sides, $AB =$ to a, $BR =$ to b, $AR =$ to $2m$. Draw $AC \parallel$ to BR and $RC \parallel$ to AB meeting at C. Draw BC. **Statement**: (?). **Proof**: (?).

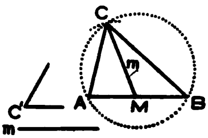
37. A side, the altitude upon it, and the angle opposite it.



Given: Side $=$ to AB, alt. $=$ to h; opposite $\angle C =$ to $\angle C'$.

Construction: Upon AB construct segment ACB which contains $\angle C =$ to $\angle C'$ (by 267). At A erect $AR \perp$ to AB and $=$ to h; etc.

38. A side, the median to it, the angle opposite it.

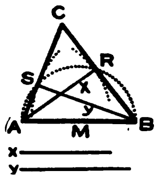


[**Statement**: $\triangle ABC$ is the required $\triangle$.]

39. One side and the altitude from its extremities to the other sides.

Given: Side $= AB$, altitudes x and y.

Construction: Bisect AB; describe a semicircle. Using A as center and x as radius, describe arc cutting the semicircle at R; etc.



40. Two sides and the altitude upon one of them.

[**Given**: Sides $=$ to AB and BC; alt. on $BC =$ to x.]

41. One side, an angle adjoining it, and the radius of the inscribed circle.

Construction: Describe $\odot$ with given radius, any center.

Construct central $\angle =$ to given $\angle$. Draw two tangents $\parallel$ to these radii; etc.

V. Construct a square, having given:

42. One side.

43. The diagonal.

44. The perimeter.

45. The sum of a diagonal and a side.

VI. Construct a **rhombus**, having given:

46. One side and an angle adjoining it.

47. One side and the altitude.

48. The diagonals.

49. One side and one diagonal. [Use 269.]

50. An angle and the diagonal to the same vertex.

51. An angle and the diagonal between two other vertices

52. One side and the radius of the inscribed circle.

VII. Construct a **rectangle**, having given:

53. Two adjoining sides.

54. A diagonal and a side.

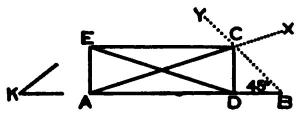
55. One side and the angle formed by the diagonals.

56. A diagonal and the sum of two adjoining sides. [See Ex. 21.]

57. A diagonal and the perimeter.

58. The perimeter and the angle formed by the diagonals.

Construction: Bisect the perimeter and take $AB =$ to half of it. Bisect $\angle K$. At A construct $\angle BAX =$ to half $\angle K$. Etc.



VIII. Construct a **parallelogram**, having given:

59. One side and the diagonals.

60. The diagonals and the angle between them.

61. One side, an angle, and the diagonal not to the same vertex.

62. One side, an angle, and the diagonal to the same vertex.

63. One side, an angle, and the altitude upon that side.

64. Two adjoining sides and the altitude.

IX. Construct an **isosceles trapezoid**, having given:

65. The bases and an angle adjoining the larger base.

66. The bases and an angle adjoining the less base.

67. The bases and the diagonal.

68. The bases and the altitude.

69. The bases and one of the equal sides.

70. One base, an angle adjoining it, and one of the equal sides.

71. One base, the altitude, and one of the equal sides.

72. One base, the radius of the circumscribed circle, and one of the equal sides. [First, describe a ⊙.]

73. One base, an angle adjoining it, and the radius of the circumscribed circle.

74. The bases and the radius of the circumscribed circle.

75. One base and the radius of the inscribed circle.

Construction: Bisect the base and erect a $\perp =$ to radius; etc.

X. Construct a **trapezoid**,* having given:

76. The bases and the angles adjoining one of them.

Construction: Take $EC =$ to longer base, and on it take $ED =$ to less base. Construct $\triangle DBC$ (by 271).

77. The four sides.

78. A base, the altitude, and the non-parallel sides.

Construction: Construct a $\triangle$ two sides of which equal the given non-$\parallel$ sides of the trapezoid, and the altitude from same vertex of which equals the given altitude. (See Ex. 34.)

79. The bases, an angle, and the altitude.

Construction: Construct ▱ on ED, having given altitude and $\angle$.

80. A base, the angles adjoining it, and the altitude.

81. The longer base, an angle adjoining it, and the non-parallel sides.

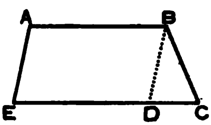
82. The shorter base, an angle not adjoining it, and the non-parallel sides.

XI. Construct the **locus** of a point which shall be:

83. At a given distance from a given point.

84. At a given distance from a given line.

* Note. It is evident that every trapezoid may be divided into a parallelogram and a triangle by drawing one line (as BD) $\parallel$ to one of the non-$\parallel$ sides. Hence the construction of a trapezoid is often merely constructing a triangle and a parallelogram.



85. At a given distance from a given circle:

(*i*) If the given radius is $<$ the given distance;
(*ii*) If the given radius is $>$ the given distance.

86. Equally distant from two given points.

87. Equally distant from two intersecting lines.

XII. Find (by intersecting loci)* the **point** *P*, which shall be:

88. At two given distances from two given points.†

89. Equally distant from three given points.

90. In a given line and equally distant from two given points.

91. In a given line and equally distant from two given intersecting lines.

92. In a given circle and equally distant from two given points.†

93. In a given circle and equally distant from two intersecting lines.†

94. Equally distant from two given intersecting lines and equally distant from two given points.†

95. At a given distance from a given line and equally distant from two given points.†

96. At a given distance from a given line and equally distant from two other intersecting lines.†

97. Equally distant from two given points and at a given distance from one of them.†

98. Equally distant from two given intersecting lines and at a given distance from one of them.†

99. At a given distance from a point and equally distant from two other points.†

100. At given distances from two given intersecting lines.†

101. At given distances from a given line and from a given circle.†

* It is well to draw the loci concerned as dotted lines. (See Ex. 105.)

† In the **Discussion**, include the answers to questions like these:

(1) Is this ever impossible? (*i.e.* must there always be such a point?)
(2) Are there ever two such points? when?
(3) Are there ever more than two? when?
(4) Is there ever only one? when?

102. At given distances from a given line and from a given point.*

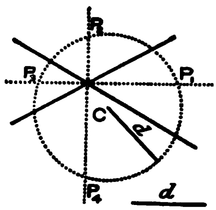
103. Equally distant from two parallels and equally distant from two intersecting lines.*

104. At a given distance from a given point and equally distant from two given parallels.*

105. At a given distance from a given point and equally distant from two given intersecting lines.

Can C be so taken that there will be no point?

Can C be so taken that there will be only one point? only two? only three? more than four?



XIII. Find (by intersecting loci) the **center** of a circle which shall:

106. Pass through three given points.†

107. Pass through a given point and touch a given line at a given point.†

108. Have a given radius and be tangent to a given line at a given point.†

109. Have a given radius, touch a given line, and pass through a given point.†

110. Pass through a given point and touch two given parallel lines.†

111. Touch two given parallels, one of them at a given point.†

112. Have a given radius and touch two given intersecting lines.†

113. Have a given radius and pass through two given points.†

114. Touch three given indefinite lines, no two of them being parallel.‡

115. Touch three given lines, only two of them being parallel.

XIV. Construct a **circle** which shall:

116. Pass through a given point and touch a given line at a given point.

117. Touch two given parallel lines, one of them at a given point.

118. Pass through a given point and touch two given parallels.

* See note (†) on preceding page.

† **Discussion**: Is this ever impossible? Are there ever two circles and hence two centers? Are there ever more than two? Etc.

‡ Four solutions. One is in 265.

119. Have a given radius, touch a given line, and pass through a given point.

120. Have its center in one line, touch another line, and have a given radius.

121. Have a given radius and touch two given intersecting lines.

122. Have a given radius and pass through two given points.

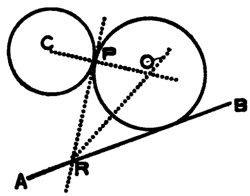
123. Have a given radius and touch a given circle at a given point. [Draw tangent to the given ⊙ at the given point.]

124. Have a given radius and touch two given circles.

125. Touch three indefinite intersecting lines.*

126. Touch two given intersecting lines, one of them at a given point.

127. Touch a given line and a given circle at a given point.



Given: Line AB; ⊙ C; point P.

Construction: Draw radius CP. Draw tangent at P meeting AB at R. Bisect $\angle PRB$, meeting CP produced at O; etc.

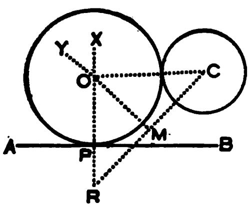
128. Be inscribed in a given sector.

Construction: Produce the radii to meet the tangent at the midpoint of the arc. In this △ inscribe a ⊙.

129. Have a given radius and touch two given circles.

130. Have a given radius, touch a given line, and a given circle.

131. Touch a given line at a given point and touch a given circle.



Given: Line AB; point P; ⊙ C.

Construction: At P erect $PX \perp$ to AB, and extend it below AB, so PR = radius ⊙ C.

Draw CR and bisect it at M.

Erect $MY \perp$ to CR at M, meeting PX at O; etc.

---

132. What is the locus of the vertices of all right triangles having the same hypotenuse?

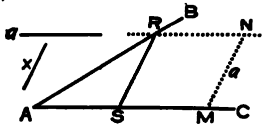
133. Through a given point on a given circumference draw two equal chords perpendicular to each other.

* Four solutions. One is in 265.

134. Draw a line of given length through a given point and terminating in two given parallels.

Construction: Use any point of one of the ‖s as center and the given length as radius to describe an arc meeting the other ‖. Join these two points. Through the given point draw a line ‖, etc.

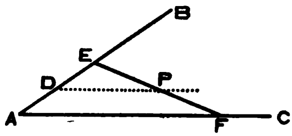
135. Draw a line, terminating in the sides of an angle, which shall be equal to one line and parallel to another.



Statement: $RS = a$, and is ‖ to x.

136. Draw a line through a given point within an angle, which is terminated by the sides of the angle and bisected by the point.

Construction: Through P draw PD ‖ to AC. Take on AB, $DE = AD$. Draw EPF; etc.

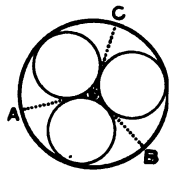


137. Circumscribe a circle about a rectangle.

138. Construct three circles having the vertices of a given triangle as centers so that each touches the other two.

Construction: Inscribe a ⊙ in the △; etc.

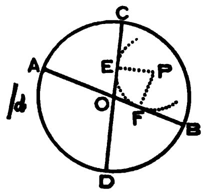
139. Construct within a circle three equal circles each of which touches the given circle and the other two.



Construction: Draw a radius, OA, and construct $\angle AOB =$ to 120° and $\angle AOC =$ to 120°. In these sectors inscribe, etc.

140. Through a point without a circle draw a secant having a given distance from the center.

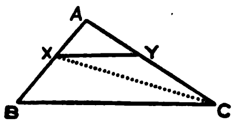
141. Draw a diameter to a circle at a given distance from a given point.



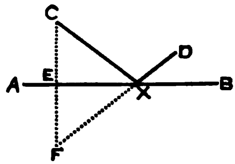
142. Through two given points within a circle draw two equal and parallel chords.

Construction: Bisect the line joining the given points and draw a diameter, etc.

143. Draw a parallel to side BC of triangle ABC, meeting AB in X and AC in Y, such that $XY = YC$.

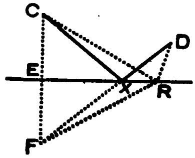


144. Find the locus of the points of contact of the tangents drawn to a series of concentric circles from an external point.



145. **Given:** Line AB and points C and D on the same side of it; find point X in AB such that $\angle AXC = \angle BXD$.

Construction: Draw $CE \perp$ to AB and produce to F so that $EF = CE$. Draw FD meeting AB in X. Draw CX.



146. Draw from one given point to another the shortest path which has one point in common with a given line.

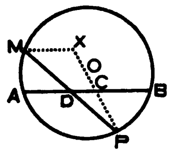
Statement: $CX + XD$ is $< CR + RD$.

Another statement of this exercise: If C is an object before a mirror ER, and D is an eye, draw a diagram showing the path of a ray of light, from C, reflected to D.

147. Draw a line parallel to side BC of triangle ABC meeting AB at X and AC at Y, so that $XY = BX + YC$.

Construction: Draw bisectors of $\angle$s B and C, meeting at O, etc.

148. Draw in a circle, through a given point of an arc, a chord that is bisected by the chord of the arc.



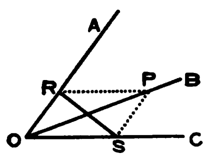
Construction: Draw radius OP meeting chord at C. Prolong PO to X so that $CX = CP$. Draw $XM \parallel$ to AB meeting $\odot$ at M. Draw PM cutting AB at D; etc. Is there any other chord from P bisected by AB?

149. Inscribe in a given circle a triangle whose angles are given.

Construction: Construct 3 central $\angle$s, doubles of the given $\angle$s. Etc.

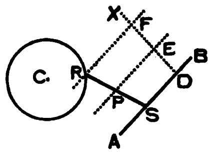
150. Circumscribe about a given circle a triangle whose angles are given.

Construction: Inscribe $\triangle$ (like Ex. 149) first, and draw tangents $\parallel$ to the sides.



151. Three lines meet in a point. Draw a line terminating in the outer two and bisected by the inner one.

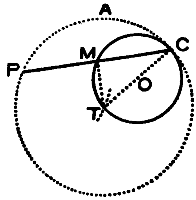
Construction: Through any point P, of OB, draw $\parallel$s to the outer lines. Draw diagonal RS; etc.



152. Draw through a given point P, a line that is terminated by a given circle and a given line and is bisected by P.

Construction: Draw any line DX meeting AB at D. Draw $PE \parallel$ to AB meeting DX at E. Take $EF = ED$; etc.

153. Through a given point without a circle draw a secant to the circle which is bisected by the circle.

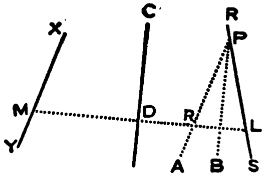


Construction: Draw arc at T, using P as center and diam. of ⊙ O as radius. Using T as center and same radius as before, describe circle touching ⊙ O at C and passing through P. Draw PC meeting ⊙ O at M.

154. Inscribe a square in a given rhombus.

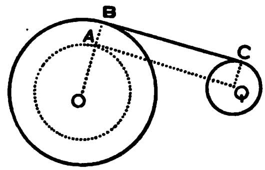
[Bisect the four ∠s formed by the diagonals.]

155. Bisect the angle formed by two lines without producing them to their point of intersection.



Construction: At P, any point in RS, draw $PA \parallel$ to XY; bisect $\angle APS$ by PB. At any point in PB erect $ML \perp$ to PB, meeting the given lines in M and L. Bisect ML at D and erect $DC \perp ML$, etc.

156. Construct a common external tangent to two circles.



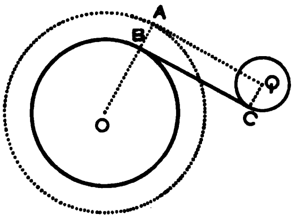
Construction: Using O as a center and a radius equal to the difference of the given radii, construct (dotted) circle. Draw QA tangent to this ⊙ from Q; draw radius OA and produce it to meet given ⊙ at B. Draw radius $QC \parallel$ to OB. Join BC.

Statement: BC is tangent to both ⊙s.

Proof: $AB = CQ$ (Const.). AB is $\parallel$ to CQ (?).

$\therefore ABCQ$ is a ▱ (?). But $\angle OAQ$ is a rt. $\angle$; etc.

157. Construct a common internal tangent to two circles.



Construction: Using O as a center and a radius equal to the sum of the given radii, construct (dotted) circle. Draw QA tangent to this ⊙ from Q. Draw radius OA meeting given ⊙ at B, etc., as above.

BOOK III

PROPORTION. SIMILAR FIGURES

274. A **ratio** is the quotient of one quantity divided by another, both being of the same kind.

275. A **proportion** is an equation whose members are ratios.

276. The **extremes** of a proportion are the first and the last terms. The **means** of a proportion are the second and the third terms.

277. The **antecedents** are the first and the third terms. The **consequents** are the second and the fourth terms.

278. A **mean proportional** is the second or the third term of a proportion in which the means are identical.

A **third proportional** is the last term of a proportion in which the means are identical.

A **fourth proportional** is the last term of a proportion in which the means are not identical.

279. A **series of equal ratios** is the equality of more than two ratios.

A **continued proportion** is a series of equal ratios in which the consequent of any ratio is the antecedent of the next following ratio.

Explanatory. A ratio is written as a fraction or as an indicated division; $\frac{a}{b}$, or $a \div b$, or $a:b$. A proportion is usually written $\frac{a}{b} = \frac{x}{y}$, or $a:b = x:y$, and is read: "a is to b as x is to y." In this proportion the extremes are a and y; the means are b and x; the antecedents are a and x; the consequents are b and y; and y is a fourth proportional to a, b, x. In the proportion $a:m = m:z$, the mean proportional is m, and z is the third proportional.

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

280. In a proportion the product of the extremes is equal to the product of the means.

Given: $\frac{a}{b}=\frac{x}{y}$ or $a:b=x:y$. **To Prove:** $ay=bx$.

Proof: $\frac{a}{b}=\frac{x}{y}$ (Hyp.). Multiply by the common denominator by and obtain, $ay=bx$ (Ax. 3).

Q.E.D.

Proposition II. Theorem

281. If the product of two quantities is equal to the product of two others, one pair may be made the extremes of a proportion and the other pair the means.

Given: $ay=bx$.

To Prove: These eight proportions:

1. $a:b=x:y$,	5. $x:y=a:b$,
2. $a:x=b:y$,	6. $x:a=y:b$,
3. $b:a=y:x$,	7. $y:x=b:a$,
4. $b:y=a:x$,	8. $y:b=x:a$.

Proof. 1. $ay=bx$ (Hyp.).

Divide each member by by, $\frac{ay}{by}=\frac{bx}{by}$ (Ax. 3).

$\therefore \frac{a}{b}=\frac{x}{y}$ or $a:b=x:y$ Q.E.D.

2. $ay=bx$ (Hyp.).

Divide by xy, etc.

3. $ay=bx$ (?).

Divide by ax, etc. Etc., etc.

Numerical Illustration. Suppose in this paragraph $a=4, b=14, x=6, y=21$; the truth of the above proportions can be clearly seen by writing these equivalents. $4\times 21=14\times 6$ (True).

1. $4:14=6:21$ (True); 2. $4:6=14:21$ (True); etc.

They will all be recognized as true proportions.

Proposition III. Theorem

282. In any proportion the terms are also in proportion by alternation (that is, the first term is to the third as the second is to the fourth).

Given: $a : b = x : y$. **To Prove:** $a : x = b : y$.

Proof:

$$a : b = x : y \qquad \text{(Hyp.)}.$$
$$\therefore\ ay = bx \qquad (280).$$
$$\therefore\ a : x = b : y \qquad (281).$$

Q.E.D.

Proposition IV. Theorem

283. In any proportion the terms are also in proportion by inversion (that is, the second term is to the first as the fourth term is to the third).

[The proof is similar to the proof of 282.]

Proposition V. Theorem

284. In any proportion the terms are also in proportion by composition (that is, the sum of the first two terms is to the first, or the second, as the sum of the last two terms is to the third, or the fourth).

Given: $a : b = x : y$. **To Prove:** $\begin{cases} a + b : a = x + y : x, \text{ or} \\ a + b : b = x + y : y. \end{cases}$

Proof: $a : b = x : y$ (Hyp.). $\therefore\ ay = bx$ (280).

Add ax to each, and obtain, $ax + ay = ax + bx$ (Ax. 2).

That is, $a(x + y) = x(a + b)$.

Hence $a + b : a = x + y : x$ (281).

Similarly, by adding by, $a + b : b = x + y : y$. Q.E.D.

Ex. 1. Is the equation $12 : 9 = 28 : 21$ a true proportion?

Ex. 2. Apply Proposition I to the above proportion.

Ex. 3. Apply Proposition III to the above proportion.

Ex. 4. Apply Proposition IV to the above proportion.

Ex. 5. Apply Proposition V to the above proportion.

Proposition VI. Theorem

285. In any proportion the terms are also in proportion by division (that is, the difference between the first two terms is to the first, or the second, as the difference between the last two terms is to the third, or the fourth).

Given: $a : b = x : y$. **To Prove:** $\begin{cases} a - b : a = x - y : x, \text{ or} \\ a - b : b = x - y : y. \end{cases}$

Proof: $a : b = x : y$ (Hyp.). $\therefore\ ay = bx$ (280).

Subtracting each side from ax, $ax - ay = ax - bx$ (Ax. 2).

That is, $a(x - y) = x(a - b)$.

Hence $a - b : a = x - y : x$ (281).

Similarly, $a - b : b = x - y : y$. Q.E.D.

Note 1. The proportions of 284 and 285 may be written in many different forms (282, 283). Thus, (1) $a \pm b : a = x \pm y : x$; (2) $a \pm b : b = x \pm y : y$; (3) $a \pm b : x \pm y = a : x$, etc.

Note 2. In any proportion the sum of the antecedents is to the sum of the consequents as either antecedent is to its consequent. Also, in any proportion the difference of the antecedents is to the difference of the consequents as either antecedent is to its consequent. (Explain.)
Thus: $a + x : b + y = a : b = x : y$. Also, $a - x : b - y = a : b = x : y$.

Proposition VII. Theorem

286. In any proportion the terms are also in proportion by composition and division (that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference).

Given: $a : b = x : y$. **To Prove:** $\dfrac{a+b}{a-b} = \dfrac{x+y}{x-y}$.

Proof: $$\frac{a+b}{a} = \frac{x+y}{x} \quad (284).$$

$$\frac{a-b}{a} = \frac{x-y}{x} \quad (285).$$

Divide the first by the second, $\dfrac{a+b}{a-b} = \dfrac{x+y}{x-y}$ (Ax. 3).

Q.E.D.

PROPOSITION VIII. THEOREM

287. In any proportion, like powers of the terms are also in proportion, and like roots of the terms are in proportion.

Given: $a : b = x : y$.

To Prove: $a^n : b^n = x^n : y^n$; and $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{x} : \sqrt[n]{y}$.

Proof: [Write the hypothesis in fractional form, etc.]

PROPOSITION IX. THEOREM

288. In two or more proportions the products of the corresponding terms are also in proportion.

Given: $a : b = x : y$, and $c : d = l : m$, and $e : f = r : s$.

To Prove: $ace : bdf = xlr : yms$.

Proof: [Write in fractional form and multiply.]

PROPOSITION X. THEOREM

289. A mean proportional is equal to the square root of the product of the extremes.

Given: $a : x = x : b$. **To Prove:** $x = \sqrt{ab}$.

Proof: [Use 280.]

PROPOSITION XI. THEOREM

290. If three terms of one proportion are equal to the corresponding three terms of another proportion, each to each, the remaining terms are also equal.

Given: $\left\{\begin{array}{l} a : b = c : m, \text{ and} \\ a : b = c : r. \end{array}\right\}$. **To Prove:** $m = r$.

Proof: $am = bc$ and $ar = bc$ (280).

$\therefore am = ar$ (Ax. 1).

$\therefore m = r$ (Ax. 3).

Q.E.D.

PROPOSITION XII. THEOREM

291. In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.

Given: $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$.

To Prove: $\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=$ etc.

Proof: Set each given ratio $=$ to m; thus,

$$\frac{a}{b}=m;\ \frac{c}{d}=m;\ \frac{e}{f}=m;\ \frac{g}{h}=m.$$

$$\therefore\ a=bm,\ c=dm,\ e=fm,\ g=hm \qquad \text{(Ax. 3).}$$

Hence, $$\frac{a+c+e+g}{b+d+f+h}=\frac{bm+dm+fm+hm}{b+d+f+h} \qquad \text{(Substitution).}$$

$$=\frac{m(b+d+f+h)}{b+d+f+h} \qquad \text{(Factoring).}$$

$$=m \qquad \text{(Canceling).}$$

$$\therefore\ \frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h} \qquad \text{(Ax. 1).}$$

Q.E.D.

Ex. 1. If $3:4=6:x$, find x. **Ex. 2.** If $8:12=12:x$, find x.

Ex. 3. Find a fourth proportional to 6, 7, and 15.

Ex. 4. Find a third proportional to 4 and 10.

Ex. 5. If $11:15=x:25$, find x. **Ex. 6.** If $4:x=x:25$, find x.

Ex. 7. Find a mean proportional between 8 and 18.

Ex. 8. If $7:x=35:48$, find x.

Ex. 9. Given, $5:8=15:24$. Write seven other true proportions containing these four numbers.

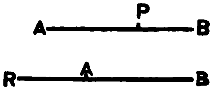
Ex. 10. If $5\times 6=2\times 15$, write eight proportions with these numbers.

Ex. 11. If $7:12=21:36$, write the proportion resulting by alternation; inversion; composition; division; composition and division.

Ex. 12. If $x+y:x-y=17:7$, write the proportions that result by virtue of composition; division; composition and division.

Ex. 13. Apply 291 to the ratios, $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}$.

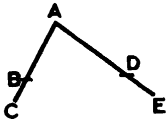
292. A **segment** of a line is any part of the line. If a line is divided by a point into two segments, they are the distances from this point of division to the extremities of the line.



The upper line AB is divided **internally** by P (a point between the extremities A and B) into segments AP and PB. The lower line AB is divided **externally** by R (a point in the prolongation of AB) into segments, RA and RB.

When a line is divided *internally*, it equals the *sum* of the segments; when it is divided *externally*, it equals the difference of the segments.

Two lines are **divided proportionally** if the ratio of the lines is equal to the ratio of the corresponding segments. Thus, AC and AE are said to be "divided proportionally" if



$$\frac{AC}{AE}=\frac{AB}{AD} \text{ or } \frac{AC}{AE}=\frac{BC}{DE} \text{ or } \frac{AC}{AE}=\frac{AB}{AD}=\frac{BC}{DE}.$$

NOTE. We have seen that it is possible to add two lines and subtract one line from another. Now it is essential that we clearly understand the significance implied by indicating the multiplication or the division of one line by another.

What is actually done is to multiply or divide the numerical measure of one line by the numerical measure of another. Thus, if one line is 8 inches long and another is 18 inches long, we say that the ratio of the first line to the second is $\frac{8}{18}$ or $\frac{4}{9}$, meaning that the smaller line is four ninths of the larger.

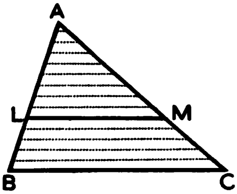
Also, in referring to the product of two lines, we merely understand that the product of their numerical measures is intended.

If a line is multiplied by itself, we obtain the square of the numerical measure of the line. The square of the line AB is written $\overline{AB}^2$ or $(AB)^2$, and the quantity that is squared is the numerical value of the length of AB.

In the preceding paragraphs of Book III, we have been considering numerical magnitudes. It should be distinctly understood that in the following geometrical propositions and demonstrations, the foregoing interpretation is implied in multiplication and division involving lines.

Historical Note. Eudoxus, one of the most prominent of the Greek mathematicians, was famous also as a physician in the fourth century B.C. One of his principal contributions to geometry was the perfection of a rigorous theory of ratio and proportion.

Proposition XIII. Theorem



293. A line parallel to one side of a triangle divides the other sides into proportional segments.

Given: $\triangle ABC$ and line $LM \parallel$ to BC.

To Prove: $AL : LB = AM : MC$.

Proof: I. If the parts AL and LB are **commensurable.**

There exists a common unit of measure of AL and LB (225). Suppose this is contained 9 times in AL and 5 times in LB.

Then $$\frac{AL}{LB} = \frac{9}{5} \qquad \text{(Ax. 3).}$$

Draw lines through the several points of division $\parallel$ to BC.

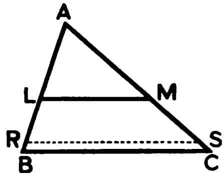
These divide AM into 9 parts and MC into 5 parts.

All these 14 parts are equal (140).

$$\therefore \frac{AM}{MC} = \frac{9}{5} \qquad \text{(Ax. 3).}$$

$$\therefore \frac{AL}{LB} = \frac{AM}{MC} \qquad \text{(Ax. 1).}$$

Q.E.D.



II. If the parts AL and LB are **incommensurable.**

There does not exist a common unit (225).

Divide AL into several equal parts (by 261).

Apply one of these as a unit of measure to LB. There is remainder, RB (225).

Draw $RS \parallel$ to BC.

Now $$\frac{AL}{LR} = \frac{AM}{MS} \qquad \text{(Case I).}$$

Indefinitely increase the *number* of equal parts of AL. That is, indefinitely decrease each part, the unit or divisor. Hence, the remainder, RB, is indefinitely decreased. (Because the remainder is $<$ the divisor.)

That is, RB approaches zero as a limit.
Also SC approaches zero as a limit.

$\therefore$ LR approaches LB as a limit (227).

Also MS approaches MC as a limit (227).

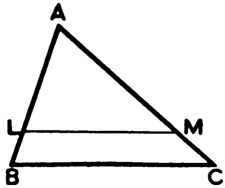
$\therefore \frac{AL}{LR}$ approaches $\frac{AL}{LB}$ as a limit.

Also $\frac{AM}{MS}$ approaches $\frac{AM}{MC}$ as a limit.

Consequently, $\frac{AL}{LB} = \frac{AM}{MC}$ (229). Q.E.D.

Proposition XIV. Theorem

294. If a line parallel to one side of a triangle intersects the other sides, it divides these sides proportionally.



Given: $\triangle ABC$; $LM \parallel$ to BC.

To Prove:

I. $AB : AC = AL : AM$.

II. $AB : AC = LB : MC$.

Proof: $AL : LB = AM : MC$ (293).

$\therefore AL + LB : AL = AM + MC : AM$ (284).

Also $AL + LB : LB = AM + MC : MC$ (284).

But $AL + LB = AB$, and $AM + MC = AC$ (Ax. 4).

Substituting, in last two proportions:

$AB : AL = AC : AM$ (Ax. 6).

Also $AB : LB = AC : MC$ (Ax. 6).

$\therefore$ I. $AB : AC = AL : AM$ (282).

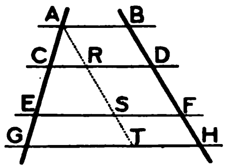
Also II. $AB : AC = LB : MC$ (282). Q.E.D.

These proportions may be combined thus: $\frac{AB}{AC} = \frac{AL}{AM} = \frac{LB}{MC}$.

Each of the above proportions may be written in eight different ways.

Ex. If, in figure of 294, AB is 9 units, AC 12 units, and AL 6 units, find AM, LB, and MC.

Proposition XV. Theorem



295. Three or more parallels intercept proportional segments on two transversals.

Given: (?).

To Prove:

$$AC : BD = CE : DF = EG : FH.$$

Proof: Draw from A, $AT \parallel$ to BH intersecting the $\parallel$s, etc.

In $\triangle AES$, $\dfrac{AE}{AS} = \dfrac{AC}{AR} = \dfrac{CE}{RS}$ (294).

In $\triangle AGT$, $\dfrac{AE}{AS} = \dfrac{EG}{ST}$ (294).

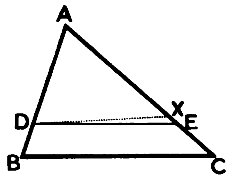
$\therefore \dfrac{AC}{AR} = \dfrac{CE}{RS} = \dfrac{EG}{ST}$ (Ax. 1).

But $AR = BD$, $RS = DF$, $ST = FH$ (124).

Hence $AC : BD = CE : DF = EG : FH$ (Ax. 6).

Q.E.D.

Proposition XVI. Theorem



296. If a line divides two sides of a triangle proportionally, it is parallel to the third side.

Given: $\triangle ABC$; line DE; the proportion $AB : AC = AD : AE$.

To Prove: DE is $\parallel$ to BC.

Proof: Through D draw $DX \parallel$ to BC meeting AC at X.

$\therefore AB : AC = AD : AX$ (294).

But $AB : AC = AD : AE$ (Hyp.).

$\therefore AX = AE$ (290).

$\therefore DX$ and DE coincide (39).

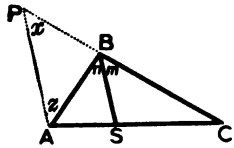
That is, DE is $\parallel$ to BC Q.E.D.

Ex. 1. Prove, as 296 is proved, that the line bisecting two sides of a triangle is parallel to the third side.

Ex. 2. In fig. of Proposition XIV, if AL is $\frac{78}{94}$ of AB, what is true of AM? of MC?

PROPOSITION XVII. THEOREM

297. The bisector of an angle of a triangle divides the opposite side into segments that are proportional to the other two sides.



Given: $\triangle ABC$; BS the bisector of $\angle ABC$.

To Prove: $AS : SC = AB : BC$.

Proof: Through A draw $AP \parallel$ to BS, meeting CB, produced, at P.

Then, in $\triangle PAC$, $AS : SC = PB : BC$ (294).

Now $\angle m = \angle x$ (67).

And $\angle n = \angle z$ (66).

But $\angle m = \angle n$ (Hyp.).

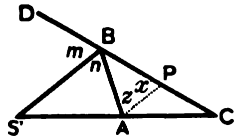
$\therefore \angle x = \angle z$ (Ax. 1).

$\therefore PB = AB$ (114).

Substituting above, $AS : SC = AB : BC$ (Ax. 6). Q.E.D.

PROPOSITION XVIII. THEOREM

298. The bisector of an exterior angle of a triangle divides the opposite side (externally) into segments that are proportional to the other two sides.



Given: $\triangle ABC$; BS', the bisector of exterior $\angle ABD$, meeting AC (externally) at S'.

To Prove: $S'A : S'C = AB : BC$.

Proof: Through A draw $AP \parallel$ to BS' meeting BC at P.

Then, in $\triangle CBS'$, $S'A : S'C = BP : BC$ (294).

Now $\angle m = \angle x$ (67).

And $\angle n = \angle z$ (66).

But $\angle m = \angle n$ (Hyp.).

$\therefore \angle x = \angle z$ (Ax. 1).

$\therefore BP = AB$ (114).

Substituting above, $S'A : S'C = AB : BC$ (Ax. 6). Q.E.D.

Ex. 1. If, in 297, $AS = 3$, $AB = 4$, $BC = 9$, find SC.

Ex. 2. If, in 297, $AC = 20$, $AB = 9$, $BC = 21$, find AS and SC.

Ex. 3. If, in 298, $S'A = 10$, $AB = 7$, $BC = 16$, find $S'C$ and AC.

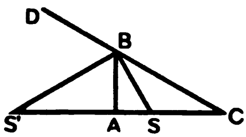
Ex. 4. If, in 298, $AC = 14$, $AB = 12$, $BC = 19$, find $S'A$ and $S'C$.

299. A line is **divided harmonically** if it is divided internally and externally in the same ratio.

In 297, the line AC is divided *internally* by S, in the ratio $AB : BC$.
In 298, the line AC is divided *externally* by S', in the ratio $AB : BC$.

Proposition XIX. Theorem

300. The bisectors of the interior and exterior angles of a triangle (at a vertex) divide the opposite side harmonically.



Given: $\triangle ABC$; BS bisecting $\angle ABC$; and BS' bisecting $\angle ABD$.

To Prove: $AS : SC = S'A : S'C$.

Proof:

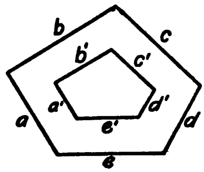
$$\frac{AS}{SC} = \frac{AB}{BC} \qquad (297).$$

$$\frac{S'A}{S'C} = \frac{AB}{BC} \qquad (298).$$

$$\therefore \frac{AS}{SC} = \frac{S'A}{S'C} \qquad (\text{Ax. } 1).$$

Q.E.D.

301. Similar polygons are polygons that are mutually equiangular and the homologous sides of which are proportional. That is, every pair of homologous angles are equal; and the ratio of one pair of homologous sides is equal to the ratio of every other pair of homologous sides,



$$a : a' = b : b' = c : c' = d : d' = \text{etc.}$$

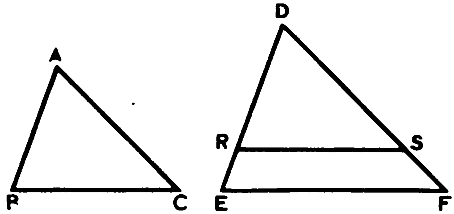
Triangles are **similar** if they are mutually equiangular and their homologous sides are proportional.

PROPOSITION XX. THEOREM

302. Two triangles are similar if they are mutually equiangular.

Given: $\triangle ABC, DEF$; $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

To Prove: The $\triangle$ are similar (that is, their sides are proportional).



Proof: Place $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and $\triangle ABC$ takes the position of $\triangle DRS$.

Then $\angle DRS = \angle E$ (Hyp.).

$\therefore RS$ is $\parallel$ to EF (71).

$\therefore DE : DR = DF : DS$ (294).

That is, $DE : AB = DF : AC$ (Ax. 6).

Likewise, by placing $\triangle ABC$ upon $\triangle DEF$ so that $\angle B$ coincides with its equal, $\angle E$, we may prove that

$$DE : AB = EF : BC$$

$\therefore DE : AB = DF : AC = EF : BC$ (Ax. 1).

$\therefore$ the $\triangle$ are similar (301).

Q.E.D.

303. COROLLARY. **Two triangles are similar if two angles of one are equal to two angles of the other.** (111 and 302.)

304. COROLLARY. **Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.** (303.)

Ex. 1. If two transversals intersect between two parallels, the triangles formed are similar.

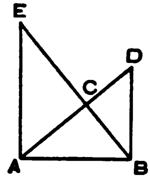
Ex. 2. Two isosceles triangles are similar if a base angle of one is equal to a base angle of the other.

Ex. 3. Two isosceles triangles are similar if the vertex angle of one is equal to the vertex angle of the other.

Ex. 4. The line joining the midpoints of two sides of a triangle forms a triangle similar to the original triangle.

Ex. 5. The diagonals of a trapezoid form, with the parallel sides, two similar triangles.

Ex. 6. If at the extremities of the hypotenuse of a right triangle perpendiculars are erected meeting the legs produced, the new triangles formed are similar.



Ex. 7. In the figure of Ex. 6, prove:

(1) Triangle ABC similar to each of the triangles ACE and BCD.

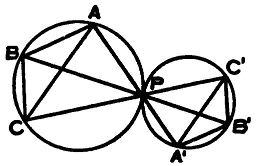
(2) Triangle ABE similar to triangle ABD.

(3) Triangle ACE similar to triangle ABD.

(4) Triangle BCD similar to triangle ABE.

(5) Triangles ABC, ABD, ABE similar.

Ex. 8. Two circles are tangent externally at P; through P three lines are drawn, meeting one circumference in A, B, C, and the other in A', B', C'. The triangles ABC and $A'B'C'$ are similar.



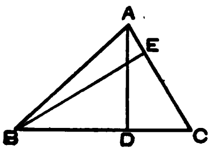
Ex. 9. Prove the same theorem if the circles are tangent internally.

Ex. 10. If two circles are tangent externally at P, and BB', CC' are drawn through P, terminating in the circumferences, the triangles PBC and $PB'C'$ are similar.

[Draw the common tangent at P.]

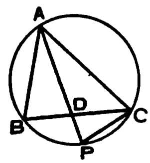
Ex. 11. Prove the same theorem if the circles are tangent internally.

Ex. 12. If AD and BE are two altitudes of triangle ABC, the triangles ACD and BCE are similar.



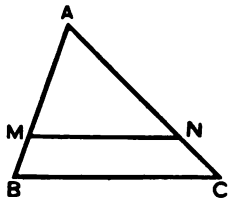
Ex. 13. If AD and BE are two altitudes of triangle ABC, meeting at O, the triangles BOD and AOE are similar.

Ex. 14. Triangle ABC is inscribed in a circle and AP is drawn to P, the midpoint of arc BC, meeting chord CB at D. The triangles ABD and ACP are similar.



PROPOSITION XXI. THEOREM

305. If a line parallel to one side of a triangle intersects the other sides, the triangle formed is similar to the original triangle.



Given: $MN \parallel$ to BC in $\triangle ABC$.

To Prove: $\triangle AMN$ similar to $\triangle ABC$.

Proof: In the $\triangle$s AMN and ABC,

$\angle A = \angle A$ (Ident.).

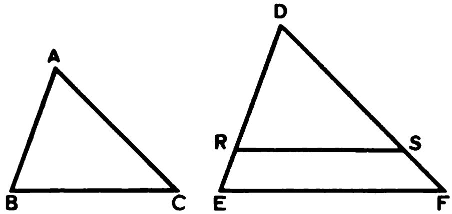
$\angle AMN = \angle B$ (67).

$\therefore \triangle AMN$ is similar to $\triangle ABC$ (303).

Q.E.D.

PROPOSITION XXII. THEOREM

306. If two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar.



Given: $\triangle$s ABC and DEF; $\angle A = \angle D$; $DE : AB = DF : AC$.

To Prove: The $\triangle$s similar.

Proof: Superpose $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and $\triangle ABC$ takes the position of $\triangle DRS$.

Now $DE : DR = DF : DS$ (Hyp.).

$\therefore RS$ is $\parallel$ to EF (296).

$\therefore \triangle DRS$ is similar to $\triangle DEF$ (305).

Substituting, $\triangle ABC$ is similar to $\triangle DEF$ (Ax. 6).

Q.E.D.

307. Corollary. If two triangles are similar to the same triangle, they are similar to each other.

Proof: The three angles of each of the first two triangles are respectively equal to the three angles of the third (301).

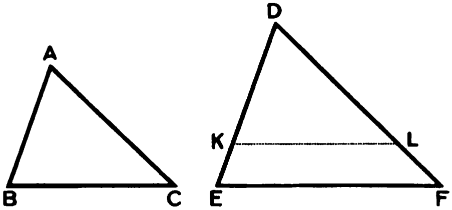
Hence the first two ▵ are mutually equiangular (Ax. 1).

Therefore they are similar (302).

Q.E.D.

Proposition XXIII. Theorem

308. If two triangles have their homologous sides proportional, they are similar.



Given: ▵ ABC and DEF; $DE : AB = DF : AC = EF : BC$.

To Prove: $\triangle\, ABC$ similar to $\triangle\, DEF$.

Proof: On DE take $DK =$ to AB; and on DF take $DL =$ to AC. Draw KL.

1st Now	$DE : AB = DF : AC$	(Hyp.).
	$DE : DK = DF : DL$	(Ax. 6).
	$\therefore\ KL$ is $\parallel$ to EF	(296).
	$\triangle\, DKL$ is similar to $\triangle\, DEF$	(305).
2d	$DE : DK = EF : KL$ (Def. sim. ▵)	(301).
Substituting,	$DE : AB = EF : KL$	(Ax. 6).
But	$DE : AB = EF : BC$	(Hyp.).
	$\therefore\ BC = KL$	(290).
3d	$\triangle\, ABC$ is congruent to $\triangle\, DKL$	(78).

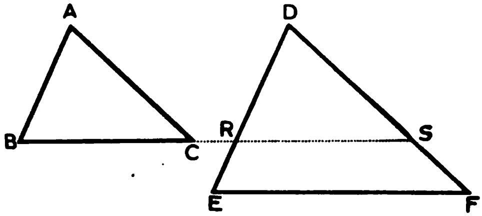
But $\triangle\, DKL$ has been proved similar to $\triangle\, DEF$.

Substituting, $\triangle\, ABC$ is similar to $\triangle\, DEF$ (Ax. 6).

Q.E.D.

PROPOSITION XXIV. THEOREM

309. If two triangles have their homologous sides parallel, they are similar.



Given: ⊿s ABC and DEF; $AB \parallel$ to DE; $AC \parallel$ to DF; and $BC \parallel$ to EF.

To Prove: $\triangle ABC$ similar to $\triangle DEF$.

Proof: Produce BC of $\triangle ABC$ until it intersects two sides of $\triangle DEF$ at R and S.

Now $\angle B = \angle DRS$, and $\angle DRS = \angle E$ (67).

$\therefore \angle B = \angle E$ (Ax. 1).

Similarly, we may prove $\angle ACB =$ to $\angle F$.

$\therefore \triangle ABC$ is similar to $\triangle DEF$ (303).

Q.E.D.

Ex. 1. Draw the figure and prove Proposition XXIV:

(*a*) If one triangle is entirely within the other;

(*b*) If no side of either, when prolonged, meets any side of the other;

(*c*) If one side of one intersects two sides of the other, without prolongation;

(*d*) If a vertex of one is upon a side of the other.

Ex. 2. If in a right triangle, a perpendicular is drawn from the vertex of the right angle upon the hypotenuse, the two new right triangles are similar to the original triangle and to each other.

Ex. 3. Are all equilateral triangles similar? Why?

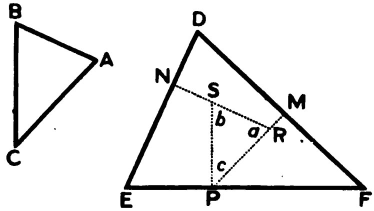
Ex. 4. Are all squares similar? all rectangles? Why?

Ex. 5. Do a square and a rectangle fulfill either of the conditions for similar polygons?

Ex. 6. If from any point in a leg of a right triangle a line is drawn perpendicular to the hypotenuse, the triangles are similar.

PROPOSITION XXV. THEOREM

310. If two triangles have their homologous sides perpendicular, they are similar.



Given: $\triangle$ *ABC* and *DEF*; *AB* $\perp$ to *DE*; *AC* $\perp$ to *DF*; etc.

To Prove: $\triangle$ *ABC* similar to $\triangle$ *DEF*.

Proof: Through *P*, any point in *EF*, construct *PR* $\parallel$ to *AC*, meeting *DF* at *M*. At *R*, any point in *PM*, draw *RS* $\parallel$ to *AB*, meeting *ED* at *N*. Draw *PS* $\parallel$ to *BC*, meeting *NR* at *S*, forming the $\triangle$ *PRS*. Now *PM* is $\perp$ to *DF* and *RN* is $\perp$ to *DE* (64).

In quadrilateral *DMRN*,

$$\angle D + \angle M + \angle MRN + \angle N = 4 \text{ rt. } \angle s \qquad (156).$$

But $$\angle M + \angle N = 2 \text{ rt. } \angle s \qquad (16).$$

$$\therefore \angle D + \angle MRN = 2 \text{ rt. } \angle s \qquad (\text{Ax. } 2).$$

But $$\angle a + \angle MRN = 2 \text{ rt. } \angle s \qquad (46).$$

$$\therefore \angle D = \angle a \qquad (49).$$

Similarly, by quadrilateral *EPSN*, it may be proved that

$$\angle E = \angle b.$$

$$\therefore \triangle DEF \text{ is similar to } \triangle PRS \qquad (303).$$

But $$\triangle ABC \text{ is similar to } \triangle PRS \qquad (309).$$

$$\therefore \triangle ABC \text{ is similar to } \triangle DEF \qquad (307).$$

Q.E.D.

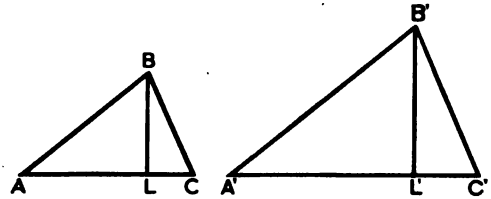
Historical Note. Thales, as early as the sixth century B.C., used similar triangles to determine the height of the great Egyptian pyramid. He measured the shadow of a pole of known height, and the shadow of the pyramid at the same time, to obtain homologous sides of similar right triangles.

PROPOSITION XXVI. THEOREM

311. Two homologous altitudes of two similar triangles are proportional to any two homologous sides.

Given: (?).

To Prove: $\frac{BL}{B'L'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$.



Proof: $\triangle ABC$ is similar to $\triangle A'B'C'$ (Hyp.).

$\therefore \angle A=\angle A'$ (301).

$\therefore \triangle ABL$ is similar to $\triangle A'B'L'$ (304).

$\therefore \frac{BL}{B'L'}=\frac{AB}{A'B'}$ (301).

But $\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$ (301).

$\therefore \frac{BL}{B'L'}=\frac{AB}{A'B'}=\frac{AC}{A'C'}=\frac{BC}{B'C'}$ (Ax. 1).

Q.E.D.

312. In similar figures, homologous angles are equal (Def.).

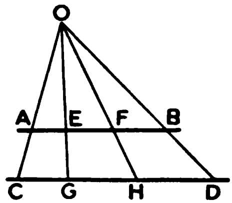
313. In similar figures, homologous sides are proportional (Def.).

The antecedents of this proportion belong to one of the similar figures, and the consequents to the other.

314. In similar triangles, homologous sides are opposite homologous angles.

Shortest sides are homologous. (Opposite smallest $\angle$s).
Medium sides are homologous. (Opposite medium $\angle$s).
Longest sides are homologous. (Opposite largest $\angle$s).

Proposition XXVII. Theorem



315. If two parallel lines are cut by three or more transversals that meet at a point, the corresponding segments of the parallels are proportional.

Given: (?).

To Prove: $\frac{AE}{CG}=\frac{EF}{GH}=\frac{FB}{HD}$.

Proof: In $\triangle OCG$, AE is $\parallel$ to CG (Hyp.).

$\therefore \triangle OAE$ is similar to $\triangle OCG$ (305).

Likewise, $\triangle OEF$ is similar to $\triangle OGH$, and $\triangle OFB$ is similar to $\triangle OHD$ (305).

$$\therefore \frac{AE}{CG}=\frac{OE}{OG}, \text{ and } \frac{EF}{GH}=\frac{OE}{OG} \quad (313).$$

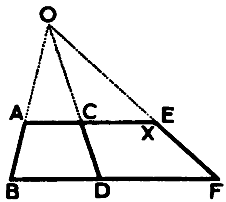
$$\therefore \frac{AE}{CG}=\frac{EF}{GH} \quad (\text{Ax. } 1).$$

Likewise $$\frac{EF}{GH}=\frac{OF}{OH}, \text{ and } \frac{OF}{OH}=\frac{FB}{HD} \quad (313).$$

$$\therefore \frac{AE}{CG}=\frac{EF}{GH}=\frac{FB}{HD} \quad (\text{Ax. } 1).$$

Q.E.D.

Proposition XXVIII. Theorem



316. If three or more non-parallel transversals intercept proportional segments on two parallels, they meet at a point. [Converse.]

Given: Transversals AB, CD, EF; parallels AE and BF; proportion, $AC : BD = CE : DF$.

To Prove: AB, CD, EF meet at a point.

Proof: Produce AB and CD until they meet, at O.

Draw OF cutting AE at X.

Now $AC : BD = CX : DF$ (315).

But $AC : BD = CE : DF$ (Hyp.).

$\therefore CX = CE$ (290).

$\therefore FE$ and FX coincide (39).

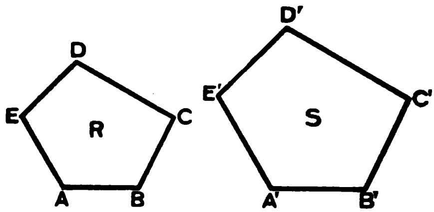
That is, FE prolonged, passes through O. Q.E.D.

Ex. 1. Show the truth of Proposition XXVIII, by an accurate diagram.

Ex. 2. In the figure of Proposition XXVIII, what is point X?

Proposition XXIX. Theorem

317. The perimeters of two similar polygons are to each other as any two homologous sides.



Given: Polygon R whose perimeter $= P$ and similar polygon S whose perimeter $= P'$.

To Prove: $P : P' = AB : A'B' = BC : B'C' =$ etc.

Proof: $AB : A'B' = BC : B'C' = CD : C'D' =$ etc. (313).

$$\therefore \frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \cdots} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \cdots \text{ etc.} \quad (291).$$

Substituting, $$\frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \text{etc.} \quad (\text{Ax. } 6).$$

Q.E.D.

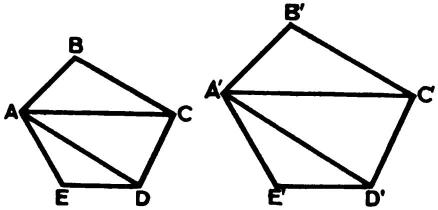
Ex. 1. The sides of a certain polygon are 4, 10, 12, 15, and 18. The shortest side of a similar polygon is 6. Find the other four sides.

Ex. 2. The perimeters of two similar polygons are 125 and 275 respectively. The longest side of the smaller polygon is 40. Find the longest side of the larger polygon.

Ex. 3. Enumerate the ways of proving two triangles similar. Which is the easiest of these ways?

Proposition XXX. Theorem

318. If two polygons are similar, they may be decomposed into the same number of triangles, similar each to each and similarly placed.



Given: Similar polygons BE and $B'E'$.

To Prove: $\triangle ABC$ similar to $\triangle A'B'C'$;
$\triangle ACD$ similar to $\triangle A'C'D'$;
$\triangle AED$ similar to $\triangle A'E'D'$.

Proof: First. $AB : A'B' = BC : B'C'$ (313).
Also $\angle B = \angle B'$ (312).
Therefore $\triangle ABC$ is similar to $\triangle A'B'C'$ (306).

Second. In $\triangle\triangle$ ABC and $A'B'C'$, $\dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$ (313).

In the similar polygons, $\dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$ (313).

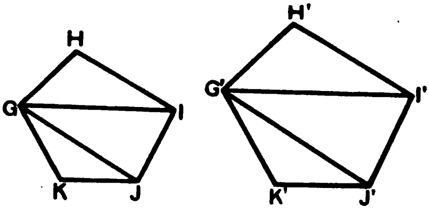
Consequently $\dfrac{AC}{A'C'} = \dfrac{CD}{C'D'}$ (Ax. 1).

In the polygons, $\angle BCD = \angle B'C'D'$ }
In the $\triangle\triangle$ ABC and $A'B'C'$, $\angle BCA = \angle B'C'A'$ } (312).
Hence, by subtraction, $\angle ACD = \angle A'C'D'$ (Ax. 2).
Therefore $\triangle ACD$ is similar to $\triangle A'C'D'$ (306).
Third. $\triangle AED$ is proved similar to $\triangle A'E'D'$ in like manner. Q.E.D.

Ex. A map is drawn on a scale of 1 in. to 400 miles. How far is Paris from Antwerp if they are $1\frac{1}{8}$ in. apart on the map?

PROPOSITION XXXI. THEOREM

319. If two polygons are composed of triangles similar each to each and similarly placed, the polygons are similar. [Converse.]



Given: $\triangle GHI$ similar to $\triangle G'H'I'$;
$\triangle GIJ$ similar to $\triangle G'I'J'$;
$\triangle GJK$ similar to $\triangle G'J'K'$.

To Prove: The polygons HK and $H'K'$ similar.

Proof: First. In $\triangle$s HGI and $H'G'I'$, $\angle H = \angle H'$ (312).

Also in these $\triangle$s $\angle HIG = \angle H'I'G'$ (312).

In $\triangle$s GIJ and $G'I'J'$, $\underline{\angle GIJ = \angle G'I'J'}$ (312).

Adding, $\angle HIJ = \angle H'I'J'$ (Ax. 2).

Likewise $\angle IJK = \angle I'J'K'$; etc.

That is, the polygons are mutually equiangular.

Second. In $\triangle$s GHI and $G'H'I'$, $\dfrac{GH}{G'H'} = \dfrac{HI}{H'I'} = \dfrac{GI}{G'I'}$ (313).

In $\triangle$s GIJ and $G'I'J'$, $\dfrac{GI}{G'I'} = \dfrac{IJ}{I'J'} = \dfrac{GJ}{G'J'}$ (?).

In $\triangle$s GJK and $G'J'K'$, $\dfrac{GJ}{G'J'} = \dfrac{JK}{J'K'} = \dfrac{KG}{K'G'}$ (?).

$$\therefore \frac{GH}{G'H'} = \frac{HI}{H'I'} = \frac{IJ}{I'J'} = \frac{JK}{J'K'} = \frac{KG}{K'G'} \qquad \text{(Ax. 1)}.$$

That is, the homologous sides are proportional.

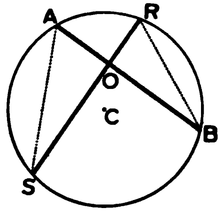
$\therefore$ The polygons are similar (301).

Q.E.D.

Ex. If two parallelograms are similar, and a diagonal in each is drawn, are the resulting triangles necessarily similar in pairs?

PROPOSITION XXXII. THEOREM

320. If through a fixed point within a circle two chords are drawn, the product of the segments of one equals the product of the segments of the other.



Given: Point O in circle C; chords AB and RS intersecting at O. (Review the note, p. 149.)

To Prove: $AO \cdot OB = RO \cdot OS$.

Proof: Draw AS and RB.

In $\triangle$ AOS and ROB, $\angle S = \angle B$ (239).

And $\angle A = \angle R$ (?).

$\therefore$ these $\triangle$ are similar (303).

$\therefore AO : RO = OS : OB$ (313).

$\therefore AO \cdot OB = RO \cdot OS$ (280). Q.E.D.

321. COROLLARY. **The product of the segments of any chord drawn through a fixed point within a circle is constant for all chords through this point.**

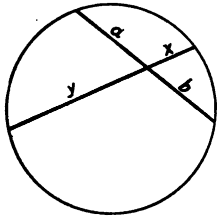
322. Direct proportion and reciprocal (or **inverse**) **proportion.**

Illustrations. I. If a man earns \$4½ each day, in 8 days he earns \$36. In 12 days he earns \$54. Hence, 8 da. : 12 da. = \$36 : \$54 is a proportion in which the antecedents belong to the same condition or circumstance, and the consequents belong to some other condition or circumstance. This is called a *direct proportion.*

II. If one man can build a certain wall in 120 days, 8 men can build it in 15 days; or 12 men in 10 days. Hence, 8 men : 12 men = 10 da. : 15 da. is a proportion in which the means belong to the same condition or circumstance, and the extremes belong to some other condition or circumstance. This is called a *reciprocal* (or *inverse*) *proportion.*

Definitions. A **direct proportion** is a proportion in which the antecedents belong to the same circumstance or figure, and the consequents belong to some other circumstance or figure. Thus, the ordinary proportions derived from similar figures are direct proportions.

A reciprocal (or **inverse**) **proportion** is a proportion in which the means belong to the same circumstance or figure, and the extremes belong to some other circumstance or figure.



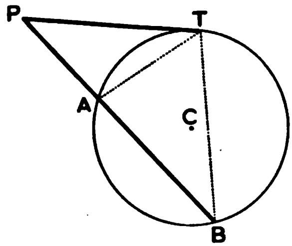
Thus, in the adjoining figure, $a \cdot b = x \cdot y$ (320). $\therefore a : x = y : b$ (281). This is a reciprocal proportion because the means are parts of *one* chord, and the extremes are parts of the *other* chord.

323. Theorem. **If through a fixed point within a circle two chords are drawn, their four segments are reciprocally** (or **inversely**) **proportional.**

Proof: [Identical with 320; omitting the last step.]

Proposition XXXIII. Theorem

324. If from a fixed point without a circle a secant and a tangent are drawn, the product of the whole secant and the external segment equals the square of the tangent.



Given: $\odot C$; secant PAB; tangent PT.

To Prove: $PB \cdot PA = \overline{PT}^2$.

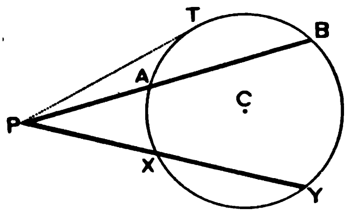
Proof: Draw AT and BT.

In $\triangle$ PAT and PBT,	$\angle P = \angle P$	(Iden.).
	$\angle PTA$ is measured by $\frac{1}{2}$ arc AT	(241).
	$\angle B$ is measured by $\frac{1}{2}$ arc AT	(236).
	$\therefore \angle PTA = \angle B$	(237).
Therefore	$\triangle PAT$ is similar to $\triangle PBT$	(303).
Hence	$PB : PT = PT : PA$	(313).
	$\therefore PB \cdot PA = \overline{PT}^2$	(280).

Q.E.D.

325. Corollary. If from a fixed point without a circle any secant is drawn, the product of the secant and its external segment is constant for all secants.

Proof: Any secant $\times$ ext. seg. $=$ (tan.)^2 $=$ constant.



326. Corollary. If from a fixed point without a circle two secants are drawn, these secants and their external segments are reciprocally (or inversely) proportional.

Proof:

$$PB \cdot PA = PY \cdot PX \qquad (325).$$

$$\therefore PB : PY = PX : PA \qquad (281).$$

Q.E.D.

327. Theorem. If from a fixed point without a circle a secant and a tangent are drawn, the tangent is a mean proportional between the secant and its external segment.

Proof: [Identical with proof of 324; omitting the last step.]

Ex. 1. If $PA = 3$ in., and $PB = 12$ in., find the length of PT.

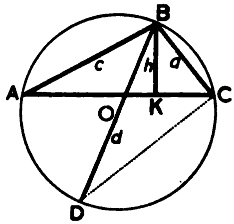
Ex. 2. If $PB = 21$ in., $PY = 15$ in., and $PA = 5$ in., find PX.

Ex. 3. Two altitudes of a triangle are reciprocally proportional to the bases to which they are drawn.

Ex. 4. If AD and BE are two altitudes of a triangle, and DE is drawn, the triangle ABC is similar to the triangle CED. [Use 306.]

Ex. 5. The four segments of the diagonals of a trapezoid are proportional.

PROPOSITION XXXIV. THEOREM



328. In any triangle the product of two sides is equal to the diameter of the circumscribed circle multiplied by the altitude upon the third side.

Given: $\triangle ABC$; circumscribed $\odot O$; altitude BK.

To Prove: $a \cdot c = d \cdot h$.

Proof: Draw chord CD. $\angle BCD =$ rt. $\angle$ (240).

In rt. $\triangle ABK$ and BCD, $\angle A = \angle D$ (239).

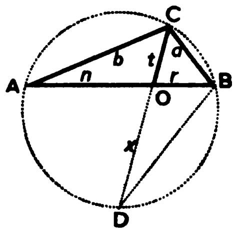
$\therefore$ these $\triangle$ are similar (304).

$\therefore c : d = h : a$ (313).

$\therefore a \cdot c = d \cdot h$ (280). Q.E.D.

PROPOSITION XXXV. THEOREM

329. In any triangle the product of two sides is equal to the square of the bisector of their included angle, plus the product of the segments of the third side formed by the bisector.



Given: $\triangle ABC$, CO bisector of $\angle ACB$.

To Prove: $a \cdot b = t^2 + n \cdot r$.

Proof: Circumscribe a $\odot$ about the $\triangle ABC$.

Produce CO to meet $\odot$ at D. Draw BD.

In $\triangle ACO$ and BCD, $\angle ACO = \angle BCD$ (Hyp.).

Also $\angle A = \angle D$ (239).

$\therefore \triangle ACO$ and BCD are similar (303).

$\therefore b : (t + x) = t : a$ (313).

$\therefore a \cdot b = t^2 + tx$ (280).

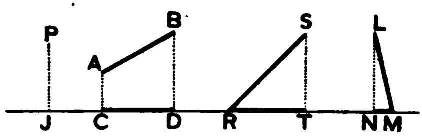
Now AB and CD are chords (Const.).

$\therefore t \cdot x = n \cdot r$ (320).

Substituting, $a \cdot b = t^2 + n \cdot r$ (Ax. 6). Q.E.D.

330. The **projection** of a **point** upon a line is the foot of the perpendicular from the point to the line.

Thus, the projection of P is J.



The **projection** of a definite **line** upon an indefinite line is the part of the indefinite line between the feet of the two perpendiculars to it, from the extremities of the definite line.

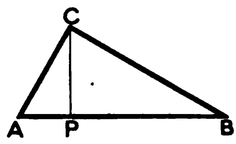
The projection of AB is CD; of RS is RT; of LM is NM.

PROPOSITION XXXVI. THEOREM

331. If in a right triangle a perpendicular is drawn from the vertex of the right angle upon the hypotenuse:

I. The triangles formed are similar to the given triangle and similar to each other.

II. The perpendicular is a mean proportional between the segments of the hypotenuse.



Given: Rt. $\triangle ABC$; $CP \perp$ to AB from C.

To Prove:

I. $\triangle$s APC, ABC, and BPC similar.

II. $AP : CP = CP : PB$.

Proof: I. In rt. $\triangle$s APC and ABC, $\angle A = \angle A$ (Iden.).

$\therefore \triangle APC$ is similar to $\triangle ABC$ (304).

In rt. $\triangle$s BPC and ABC, $\angle B = \angle B$ (Iden.).

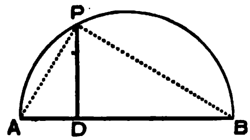
$\therefore \triangle BPC$ is similar to $\triangle ABC$ (304).

Therefore $\triangle$s APC, ABC, and BPC are all similar (307).

II. In the $\triangle$s APC and BPC, $AP : CP = CP : PB$ (313).

Q.E.D.

332. COROLLARY. **If from any point in a circumference a perpendicular is drawn to a diameter, it is a mean proportional between the segments of the diameter.**



Given: (?). **To Prove:** (?).

Proof: Draw chords AP and BP.

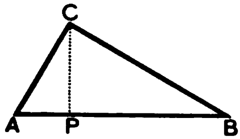
$\triangle APB$ is a rt. $\triangle$ (240).

$\therefore AD : PD = PD : DB$ (331, II).

Q.E.D.

PROPOSITION XXXVII. THEOREM

333. The square of a leg of a right triangle is equal to the product of the hypotenuse and the projection of this leg upon the hypotenuse.



Given: Rt. $\triangle ABC$; AC and BC the legs.

To Prove: I. $\overline{AC}^2 = AB \cdot AP$.

II. $\overline{BC}^2 = AB \cdot BP$.

Proof: I. The rt. $\triangle$s ABC and APC are similar (331, I).

$\therefore AB : AC = AC : AP$ (313).

$\therefore \overline{AC}^2 = AB \cdot AP$ (280).

II. The rt. $\triangle$s ABC and BCP are similar (?).

$\therefore AB : BC = BC : BP$ (313).

$\overline{BC}^2 = AB \cdot BP$ (280).

Q.E.D.

Ex. 1. If, in 331, $AP = 3$, $PB = 27$, find CP.

Ex. 2. If, in 333, $AP = 4$, $PB = 21$, find AC and BC.

Ex. 3. If, in 333, $AB = 20$, $AC = 10$, find AP, BP, CP, and BC.

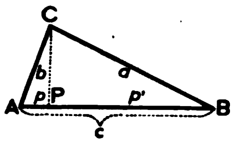
Ex. 4. If, in 331, $AP = 10$ and $CP = 20$, find BP.

Ex. 5. If, in 333, $AB = 45$ and $AC = 15$, find AP, BP, CP, and BC.

Ex. 6. If, in 333, $AP = 9$ and $BP = 16$, find AC, PC, and BC.

Ex. 7. A stone arch in the shape of the arc of a circle is 4 ft. high. The chord of half the arch is 10 ft. Find the diameter.

PROPOSITION XXXVIII. THEOREM



334. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.

Given: Rt. $\triangle\ ABC$.

To Prove: $a^2 + b^2 = c^2$.

Proof: Draw $CP \perp$ to the hypotenuse AB.

Denote AP by p and PB by p'.

Now	$b^2 = c \cdot p$	(333).
Also	$a^2 = c \cdot p'$	(?).
Adding,	$a^2 + b^2 = c \cdot p + c \cdot p'$	(Ax. 2).
	$= c(p + p')$	
	$= c \cdot c = c^2$.	Q.E.D.

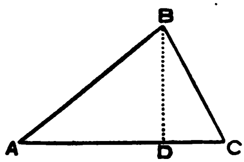
335. COROLLARY. **The square of either leg of a right triangle is equal to the square of the hypotenuse minus the square of the other leg.**

That is, $a^2 = c^2 - b^2$; and $b^2 = c^2 - a^2$ (Ax. 2).

Ex. 1. If $AC = 28$ and $BC = 45$, find AB.

Ex. 2. If $AC = 21$ and $AB = 29$, find BC.

Ex. 3. The square of the altitude of an equilateral triangle equals three fourths the square of a side.

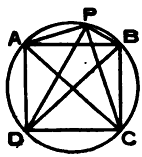


Ex. 4. In any triangle the difference of the squares of two sides is equal to the difference of the squares of their projections on the third side.

[$\overline{AB}^2 = (?)$; $\overline{BC}^2 = (?)$. Subtract, etc.]

Ex. 5. If the altitudes of triangle ABC meet at O, $\overline{AB}^2 - \overline{AC}^2 = \overline{BO}^2 - \overline{CO}^2$.

[Consult Ex. 4 and substitute.]

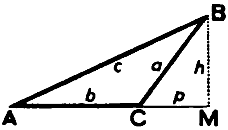


Ex. 6. If lines are drawn from any point in a circle to the four vertices of an inscribed square, the sum of the squares of these four lines is equal to twice the square of the diameter.

Proof: $\triangle$s APC, DPB are rt. $\triangle$s; etc.

PROPOSITION XXXIX. THEOREM

336. In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these two sides and the projection of the other side upon that one.



Given: Obtuse $\triangle ABC$; etc.

To Prove: $c^2 = a^2 + b^2 + 2\,bp$.

Proof: In right $\triangle CBM$, $h^2 + p^2 = a^2$ (334).

In right $\triangle ABM$, $c^2 = h^2 + (p + b)^2$ (334).

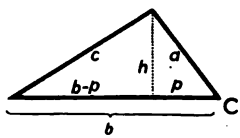
$$= \underbrace{h^2 + p^2} + b^2 + 2\,bp$$

$$= \quad a^2 \quad + b^2 + 2\,bp \qquad \text{(Ax. 6).}$$

Q.E.D.

PROPOSITION XL. THEOREM

337. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of these two sides and the projection of the other side upon that one.



Given: (?).

To Prove: $c^2 = a^2 + b^2 - 2\,bp$.

Proof: In one rt. $\triangle$, $h^2 + p^2 = a^2$ (334).

In the other rt. $\triangle$, $c^2 = h^2 + (b - p)^2$ (334).

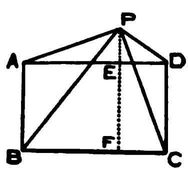
$$= \underbrace{h^2 + p^2} + b^2 - 2\,bp$$

$$= \quad a^2 \quad + b^2 - 2\,bp \qquad \text{(Ax. 6).}$$

Q.E.D.

NOTE. This theorem is equally true in the case of an obtuse triangle.

Ex. If lines are drawn from any external point to the vertices of a rectangle $ABCD$, the sum of the squares of two of them which are drawn to a pair of opposite vertices is equal to the sum of the squares of the other two.

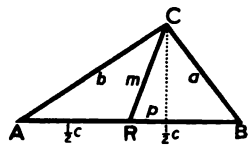


To Prove: $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.

Proposition XLI. Theorem

338. I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side.

II. The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon that side.



Given: $\triangle ABC$; median $= m$; its projection $= p$; and side $b >$ side a.

To Prove:

I. $b^2 + a^2 = 2(\frac{1}{2}c)^2 + 2m^2$.

II. $b^2 - a^2 = 2cp$.

Proof: In $\triangle s$ ARC and BRC, $AR = BR$ (Hyp.).

And $CR = CR$ (Iden.).

Also $AC > BC$ (Hyp.).

$\therefore \angle ARC > \angle BRC$ (92).

That is, $\angle ARC$ is obtuse and $\angle BRC$ is acute.

$\therefore$ In $\triangle ARC$, $b^2 = (\frac{1}{2}c)^2 + m^2 + cp$ (336).

In $\triangle BRC$, $a^2 = (\frac{1}{2}c)^2 + m^2 - cp$ (337).

I. Adding, $b^2 + a^2 = 2(\frac{1}{2}c)^2 + 2m^2$ (Ax. 2).

II. Subtracting, $b^2 - a^2 = 2cp$ (Ax. 2).

Q.E.D.

339. Formulas. If the vertices of a triangle are denoted by A, B, C, the lengths of the sides opposite are denoted by a, b, c, respectively; the altitude upon these sides by h_a, h_b, h_c, respectively; the bisectors of the angles by t_a, t_b, t_c, respectively; the medians by m_a, m_b, m_c, respectively; the segments of the sides formed by the bisectors of the opposite angles by n_a and r_a, n_b and r_b, n_c and r_c; and the projections as follows: the projection of side a upon side b, by ${}_ap_b$; of side a upon side c, by ${}_ap_c$; of side b upon side c, by ${}_bp_c$; etc.

It is assumed that a, b, c are known. The following values of the various lines in a triangle are obtained by solving the equations already established.

I. **Projections.**

1. If $\angle C$ is obtuse, ${}_ap_b = \dfrac{c^2 - a^2 - b^2}{2b}$; ${}_bp_a = \dfrac{c^2 - a^2 - b^2}{2a}$; etc.
2. If $\angle C$ is acute, ${}_ap_b = \dfrac{a^2 + b^2 - c^2}{2b}$; ${}_bp_a = \dfrac{a^2 + b^2 - c^2}{2a}$; etc.

II. **Altitudes.** $h_b = \sqrt{a^2 - {}_ap_b{}^2}$; $h_a = \sqrt{b^2 - {}_bp_a{}^2}$; etc.

III. **Medians.** $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$; $m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}$; etc.

IV. **Bisectors.** $t_c = \sqrt{ab - n_cr_c}$*; $t_a = \sqrt{bc - n_ar_a}$*; $t_b = \sqrt{ac - n_br_b}$.*

V. **Diameter** of circumscribed circle $= \dfrac{ac}{h_b}$; $= \dfrac{ab}{h_c}$; $= \dfrac{bc}{h_a}$.

VI. **Largest Angle.** Denote largest angle by C.

1. If $c^2 = a^2 + b^2$, $\angle C$ is right (334).
2. If $c^2 = a^2 + b^2$ *plus* something, $\angle C$ is obtuse (336).
3. If $c^2 = a^2 + b^2$ *minus* something, $\angle C$ is acute (337).

Ex. 1. If the sides of a triangle are $a = 7$, $b = 10$, $c = 12$, find the nature of $\angle C$.

Ex. 2. In the same triangle find m_a. Find m_b. Find m_c.

Ex. 3. In the same triangle find ${}_ap_b$. Find ${}_bp_a$. Find ${}_ap_c$. Find ${}_bp_c$. Find ${}_cp_a$. Find ${}_cp_b$.

Ex. 4. Find h_a. Find h_b. Find h_c.

Ex. 5. Find the diameter of the circumscribed circle.

Ex. 6. Find n_a and r_a. Find n_b and r_b. Find n_c and r_c.

Ex. 7. Find t_a. Find t_b. Find t_c.

CONCERNING ORIGINALS

340. First determine from the nature of each numerical exercise upon which theorem it depends, and then apply that theorem.

* The segments n and r can be found by 297; $n_b : r_b = c : a$, etc.

ORIGINAL EXERCISES (NUMERICAL)

1. The legs of a right triangle are 12 and 16 inches. Find the hypotenuse.

2. The side of a square is 6 feet. What is the diagonal?

3. The base of an isosceles triangle is 16 and the altitude is 15. Find the equal sides.

4. The tangent to a circle from a point is 12 inches and the radius of the circle is 5 inches. Find the length of the line joining the point to the center.

5. In a circle whose radius is 13 inches, what is the length of a chord 5 inches from the center?

6. The length of a chord is 2 feet and its distance from the center is 35 inches. Find the radius of the circle.

7. The hypotenuse of a right triangle is 2 feet 2 inches, and one leg is 10 inches. Find the other.

8. The base of an isosceles triangle is 90 inches and the equal sides are each 53 inches. Find the altitude.

9. The radius of a circle is 4 feet 7 inches. Find the length of the tangent drawn from a point 6 feet 1 inch from the center.

10. How long is a chord 21 yards from the center of a circle whose radius is 35 yards?

11. Each side of an equilateral triangle is 4 feet. Find the altitude.

12. The altitude of an equilateral triangle is 8 feet. Find the side.

13. Each side of an isosceles right triangle is a. Find the hypotenuse.

14. If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centers?

15. The diagonal of a rectangle is 82 and one side is 80. Find the other.

16. The length of a tangent to a circle whose diameter is 20, from an external point, is 26. What is the distance from this point to the center?

17. The diagonal of a square is 10. Find each side.

18. Find the length of a chord 2 feet from the center of a circle whose diameter is 5 feet.

19. A flagpole was broken 16 feet from the ground, and the top struck the ground 63 feet from the foot of the pole. How long was the pole?

20. The top of a ladder 17 feet long reaches a point on a wall 15 feet from the ground. How far is the lower end of the ladder from the wall?

21. A chord 2 feet long is 5 inches from the center of a circle. How far from the center is a chord 10 inches long? [Find the radius.]

22. The diameters of two concentric circles are 1 foot 10 inches and 10 feet 2 inches. Find the length of a chord of the larger which is tangent to the less.

23. The lower ends of a post and a flagpole are 42 feet apart; the post is 8 feet high and the pole, 48 feet. How far is it from the top of one to the top of the other?

24. The radii of two circles are 8 inches and 17 inches, and their centers are 41 inches apart. Find the lengths of their common external tangents; of their common internal tangents.

25. A ladder 65 feet long stands in a street. If it inclines toward one side, it will touch a house at a point 16 feet above the pavement; if to the other side, it will touch a house at a point 56 feet above the pavement. How wide is the street?

26. Two parallel chords of a circle on opposite sides of the center are 4 feet, and 40 inches long, respectively, and the distance between them is 22 inches. Find the radius of the circle.

[Draw the radii to ends of chords; these = hypotenuses = R; the distances from the center $= x$ and $22 - x$.]

27. The legs of an isosceles trapezoid are each 2 ft. 1 in. long, and one of the bases is 3 ft. 4 in. longer than the other. Find the altitude.

28. One of the non-parallel sides of a trapezoid is perpendicular to both bases, and is 63 feet long; the bases are 41 feet and 25 feet long. Find the length of the remaining side.

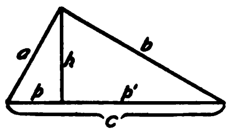
29. If $a = 10$, $h = 6$, find p, c, p', b.

30. If $h = 8$, $p' = 4$, find b, c, p, a.

31. If $a = 10$, $p' = 15$, find p, c, h, b.

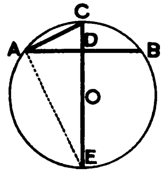
32. If $a = 9$, $b = 12$, find c, p, p', h.

33. If $p = 3$, $p' = 12$, find h, a, b.



34. The line joining the midpoint of a chord to the midpoint of its arc is 5 inches. If the chord is 2 feet long, what is the diameter?

[$\triangle ACE$ is rt. $\triangle$ (?). $\therefore \overline{AC}^2 = CE \cdot CD$ (?).]



35. If the chord of an arc is 60 and the chord of its half is 34, what is the diameter?

36. The line joining the midpoint of a chord to the midpoint of its arc is 6 inches. The chord of half this arc is 18 inches. Find the diameter. Find the length of the original chord.

37. To a circle whose radius is 10 inches, two tangents each 2 feet long are drawn from a point. Find the length of the chord joining their points of contact.

38. The sides of a triangle are 6, 9, 11. Find the segments of the shortest side made by the bisector of the opposite angle.

39. Find the segments of the longest side made by the bisector of the largest angle in Ex. 38.

40. The sides of a triangle are 5, 9, 12. Find the segments of the shortest side made by the bisector of the opposite exterior angle; also of the medium side made by the bisector of its opposite exterior angle.

41. In the figure of 295, if $AC = 3$, $CE = 5$, $EG = 8$, $BD = 4$, find DF and FH.

42. If the sides of a triangle are 6, 8, 12 and the shortest side of a similar triangle is 15, find its other sides.

43. If the homologous altitudes of two similar triangles are 9 and 15 and the base of the former is 21, what is the base of the latter?

44. In the figure of 315, $AE = 4$, $EF = 6$, $FB = 9$, $GH = 15$. Find CG and CD.

45. The sides of a pentagon are 5, 6, 8, 9, 18, and the longest side of a similar pentagon is 78. Find the other sides.

46. A pair of homologous sides of two similar polygons are 9 and 16. If the perimeter of the first is 117, what is the perimeter of the second?

47. The perimeters of two similar polygons are 72 and 120. The shortest side of the former is 4. What is the shortest side of the latter?

48. Two similar triangles have homologous bases 20 and 48. If the altitude of the latter is 36, find the altitude of the former.

49. The segments of a chord, made by a second chord, are 4 and 27. One segment of the second chord is 6. Find the other.

50. One of two intersecting chords is 19 inches long and the segments of the other are 5 inches and 12 inches. Find the segments of the first chord.

51. Two secants are drawn to a circle from a point; their lengths are 15 inches and $10\frac{1}{2}$ inches. The external segment of the latter is 10 inches. Find the external segment of the former.

52. The tangent to a circle is 1 foot long and the secant from the same point is 1 foot 6 inches. Find the chord part of the secant.

53. The internal segment of a secant 25 inches long is 16 inches. Find the tangent from the same point to the same circle.

54. Two secants to a circle from a point are $1\frac{1}{2}$ feet and 2 feet long; the tangent from the same point is 12 inches. Find the external segments of the two secants.

55. If the sides of a triangle are 5, 6, 8, is the angle opposite 8 right, acute, or obtuse? if the sides are 8, 7, 4?

56. If the sides of a triangle are 8, 9, 12, is the largest angle right, acute, or obtuse? if the sides are 13, 7, 11?

57. The sides of a triangle are x, y, z. If z is the greatest side, when will the angle opposite be right? obtuse? acute?

58. The sides of a triangle are 6, 8, 9. Find the length of the projection of side 6 upon side 8; of side 8 upon side 9; of side 9 upon side 6.

59. The sides of a triangle are 5, 6, 9. Find the length of the projection of side 6 upon side 5; of side 9 upon side 6.

60. Find the three altitudes in a triangle with sides 9, 10, 17.

61. Find the three altitudes in a triangle with sides 11, 13, 20.

62. Find the diameter of a circle circumscribed about a triangle with sides 17, 25, 26.

63. Find the length of the bisector of the least angle of a triangle with sides 7, 15, 20; also of the largest angle.

64. Find the length of the bisector of the largest angle of a triangle with sides 12, 32, 33; also of the other angles.

65. Find the three medians in a triangle with sides 4, 7, 9.

66. Find the product of the segments of every chord drawn through a point 4 units from the center of a circle whose radius is 10 units.

67. The bases of a trapezoid are 12 and 20, and the altitude is 8. The other sides are produced to meet. Find the altitude of the larger triangle formed.

68. The shadow of a yardstick perpendicular to the ground is $4\frac{1}{2}$ feet. Find the height of a tree whose shadow at the same time is 100 yards.

69. There are two belt-wheels 3 feet 8 inches and 1 foot 2 inches in diameter, respectively. Their centers are 9 feet 5 inches apart. Find the length of the belt suspended between the wheels if the belt does not cross itself; also the length of the belt if it does cross.

SUMMARY

341. Triangles are proved **similar** by showing that they have:

(1) Two angles of one equal to two angles of the other.

(2) An acute angle of one equal to an acute angle of the other. [In right triangles.]

(3) Homologous sides proportional.

(4) An angle of one equal to an angle of the other and the including sides proportional.

(5) Their sides respectively parallel or perpendicular.

Four lines are proved **proportional** by showing that they are:

(1) Homologous sides of similar triangles.

(2) Homologous sides of similar polygons.

(3) Homologous lines of similar figures.

The product of two lines is proved equal to the product of two others, by proving these four lines proportional and making the product of the extremes equal to the product of the means.

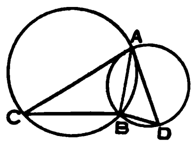
One line is proved a **mean proportional** between two others by proving that two triangles containing this line in common are similar, and obtaining the proportion from their sides.

In cases dealing with the **square of a line**, one uses:

(1) Similar triangles having this line in common, or,

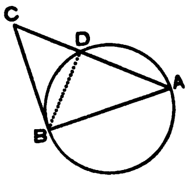
(2) A right triangle containing this line as a part.

ORIGINAL EXERCISES (THEOREMS)



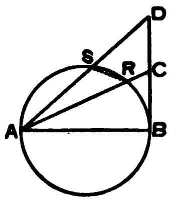
1. In any right triangle the product of the hypotenuse and the altitude upon it is equal to the product of the legs.

2. If two circles intersect at A and B, and AC and AD are drawn, each a tangent to one circle and a chord of the other, the common chord AB is a mean proportional between BC and BD.



3. If AB is a diameter and BC a tangent, and AC meets the circumference at D, the diameter is a mean proportional between AC and AD.

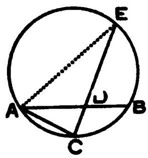
4. If two circles are tangent externally, the chords formed by a straight line drawn through their point of contact have the same ratio as the diameters of the circles.



5. If a tangent is drawn from one extremity of a diameter, meeting secants from the other extremity, these secants and their internal segments are reciprocally proportional.

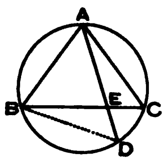
To Prove: $AC:AD = AS:AR$.

Proof: Draw RS. In ▲ ARS and ACD, $\angle A = \angle A$ and $\angle ARS = \angle D$. (Explain.) Etc.



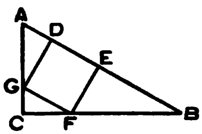
6. If AB is a chord and CE, another chord, drawn from C, the midpoint of arc AB, meeting chord AB at D, AC is a mean proportional between CD and CE.

Prove the above theorem and deduce that, $CE \cdot CD$ is constant for all positions of the point E on arc AEB



7. If chord AD is drawn from vertex A of inscribed isosceles triangle ABC, cutting BC at E, AB is a mean proportional between AD and AE.

Prove the above theorem and deduce that, $AD \cdot AE$ is constant for all positions of the point D on arc BDC.



8. If a square is inscribed in a right triangle so that one vertex is on each leg of the triangle and the other two vertices on the hypotenuse, the side of the square is a mean proportional between the other segments of the hypotenuse.

To Prove: $AD:DE = DE:EB$.

First prove ▲ ADG and BEF similar.

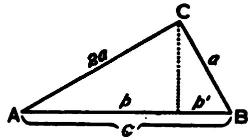
9. If the sides of two unequal triangles are respectively parallel, the lines joining homologous vertices meet in a point. (These lines to be produced if necessary.)

10. AB is any chord; AC is a tangent and CDE is a secant parallel to AB cutting the circle at D and E. Prove that $AC:AE = DC:BE$.

11. Prove theorem of 320, by drawing two other auxiliary lines.

12. Prove theorem of 316 if point O is between the parallels.

13. Prove theorem of 327 by drawing auxiliary lines AY and BX.



14. If one leg of a right triangle is double the other, its projection upon the hypotenuse is four times the projection of the other.

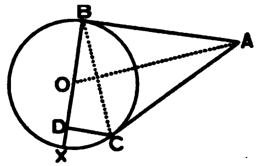
Proof: $(2\,a)^2 = cp;\ a^2 = cp'$ (?).

$$\therefore p = \frac{4\,a^2}{c};\ \ p' = \frac{a^2}{c}\ \ \text{(Ax. 3)}.\ \ \therefore p = 4\,p' \qquad (?).$$

15. If the bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles.

16. The tangents to two intersecting circles from any point in their common chord produced are equal. [Use 324.]

17. If two circles intersect, their common chord, produced, bisects their common tangents.



18. If AB and AC are tangents to a circle from A; CD is perpendicular to diameter BOX from C; then $AB \cdot CD = BD \cdot BO$.

19. If the altitude of an equilateral triangle is h, find the side.

20. If one side of a triangle is divided by a point into segments which are proportional to the other sides, a line from this point to the opposite angle bisects that angle.

To Prove: $\angle n = \angle m$ in figure of 297.

Proof: Produce CB to P, making $BP = AB$; draw AP; etc.

21. State and prove the converse of 298.

22. Two rhombuses are similar if an angle of one is equal to an angle of the other.

23. If two circles are tangent internally and any two chords of the greater are drawn from their point of contact, they are divided proportionally by the less circle.

[Draw diameter to point of contact and prove the right ▲ similar.]

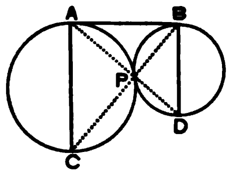
24. The non-parallel sides of a trapezoid and the line joining the midpoints of the bases, if produced, meet at a point. [Use Ax. 3 and 316.]

25. The diagonals of a trapezoid and the line joining the midpoint of the bases meet at a point.

26. If one chord bisects another, either segment of the latter is a mean proportional between the segments of the other.

27. Two parallelograms are similar if they have an angle of the one equal to an angle of the other and the including sides proportional.

28. Two rectangles are similar if two adjoining pairs of homologous sides are proportional.



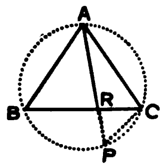
29. If two circles are tangent externally, the common exterior tangent is a mean proportional between the diameters.

[Draw chords PA, PC, PB, PD.
Prove $\angle APB$ a rt. $\angle$.
Then prove APD and BPC straight lines.
Then prove $\triangle$ ABC and ABD similar.]

30. In any rhombus the sum of the squares of the diagonals is equal to the square of half the perimeter.

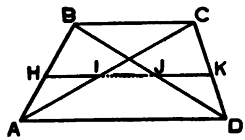
31. If in an angle a series of parallel lines are drawn having their ends in the sides of the angle, their midpoints lie in one straight line.

32. If ABC is an isosceles triangle and BX is the altitude upon AC (one of the legs), $\overline{BC}^2 = 2\,AC \cdot CX$. [Use 337.]



33. In an isosceles triangle the square of one leg is equal to the square of the line drawn from the vertex to any point of the base, plus the product of the segments of the base.

Proof: Circumscribe a $\odot$; use method of 329.



34. If a line is drawn in a trapezoid parallel to the bases, the segments between the diagonals and the non-parallel sides are equal.

Proof: $\triangle$ AHI and ABC are similar (?); $\triangle$ DKJ and DCB also.

$\therefore \dfrac{AH}{AB} = \dfrac{HI}{BC}$, and $\dfrac{DK}{DC} = \dfrac{JK}{BC}$. (Explain.)

But $\dfrac{AH}{AB} = \dfrac{DK}{DC}$ (295). $\therefore \dfrac{HI}{BC} = \dfrac{JK}{BC}$ (Axiom 1). Etc.

35. A line through the point of intersection of the diagonals of a trapezoid, and parallel to the bases, is bisected by that point.

36. If M is the midpoint of hypotenuse AB of right triangle ABC, $\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2 = 8\,\overline{CM}^2$.

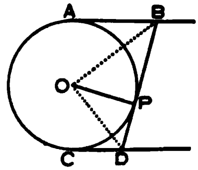
37. The squares of the legs of a right triangle have the same ratio as their projections upon the hypotenuse.

38. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares of one pair of opposite sides is equal to the sum of the squares of the other pair.

39. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals. [Use 338, I.]

40. If DE is drawn parallel to the hypotenuse AB of right triangle ABC, meeting AC at D and CB at E, $\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2$.

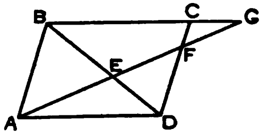
[Use 4 rt. $\triangle$ having vertex C.]



41. If between two parallel tangents a third tangent is drawn, the radius of the circle is a mean proportional between the segments of the third tangent.

To Prove: $BP : OP = OP : PD$.

Proof: $\triangle BOD$ is a rt $\triangle$ (?). Etc.

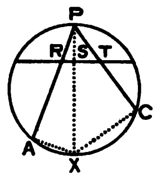


42. If $ABCD$ is a parallelogram, BD a diagonal, AG any line from A meeting BD at E, CD at F, and BC (produced) at G, AE is a mean proportional between EF and EG.

Proof: $\triangle ABE$ and EDF are similar (?); also $\triangle ADE$ and BEG (?). Obtain two ratios equal to $BE : ED$ and then apply Ax. 1.

43. An interior common tangent of two circles divides the line joining their centers into segments proportional to the radii.

44. An exterior common tangent of two circles divides the line joining their centers (externally) into segments proportional to the radii.



45. The common internal tangents of two circles and the common external tangents meet on the line determined by the centers of the circles.

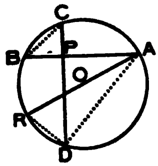
46. If from the midpoint P, of an arc subtended by a given chord, chords are drawn cutting the given chord, the product of each whole chord from P and its segment adjacent to P is constant.

Proof: Take two such chords, PA and PC; draw diameter PX; etc. Rt. $\triangle PST$ and PCX are similar. (Explain.)

47. If from any point within a triangle ABC, perpendiculars to the sides are drawn — OR to AB, OS to BC, OT to AC,

$\overline{AR}^2 + \overline{BS}^2 + \overline{CT}^2 = \overline{BR}^2 + \overline{CS}^2 + \overline{AT}^2$. [Draw OA, OB, OC.]

48. If two chords intersect within a circle and at right angles, the sum of the squares of their four segments equals the square of the diameter.

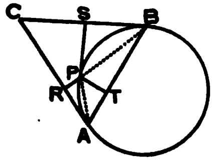


To Prove: $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = \overline{AR}^2$.

Proof: Draw BC, AD, RD. Chord BR is $\perp$ to AB (240).

$\therefore CD$ is $\parallel$ to BR (?) $\therefore$ arc BC = arc RD (?).
$\therefore$ chord BC = chord RD (?). Also $\triangle ARD$ is rt. $\triangle$ (?).
Now $\overline{AR}^2 = \overline{AD}^2 + \overline{DR}^2$ (?).
But $\overline{AD}^2 = \overline{AP}^2 + \overline{PD}^2$ and $\overline{DR}^2$ = etc.

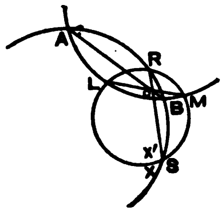
49. The perpendicular from any point of an arc upon its chord is a mean proportional between the perpendiculars from the same point to the tangents at the ends of the chord.



To Prove: $PR : PT = PT : PS$.

Proof: Prove $\triangle$ ARP and BTP similar; also $\triangle$ APT and PBS (?). Thus, get two ratios each = to $PA : PB$.

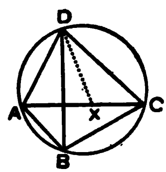
50. If each of three circles intersects the other two, the three common chords meet in a point.



To Prove: AB, LM, RS meet at O.

Proof: Suppose AB and LM meet at O. Draw RO and produce it to meet the ⓢ at X and X'. Prove $OX = OX'$ (by 320). $\therefore X$, X', S are coincident.

51. In an inscribed quadrilateral the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.



Proof: Draw DX making $\angle CDX = \angle ADB$; $\triangle$ ADB and CDX are sim. (?); also $\triangle$ BCD and ADX (?).

Hence $AB \cdot DC = DB \cdot XC$ (?),
Also $AD \cdot BC = DB \cdot AX$ (?).
Adding, etc.

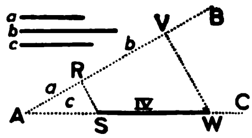
52. If AB is a diameter, BC and AD tangents, meeting chords AF and BF (produced) at C and D respectively, AB is a mean proportional between the tangents BC and AD.

53. If from a point A on the circumference of a circle two chords are drawn and a line parallel to the tangent at A meet them, the chords and their segments nearer to A are inversely proportional.

CONSTRUCTION PROBLEMS

PROPOSITION XLII. PROBLEM

342. To find a fourth proportional to three given lines.



Given: Three lines a, b, c.

Required: To find a fourth proportional to a, b, c.

Construction: Take two indefinite lines, AB and AC, meeting at A. On AB take $AR =$ to a, $RV =$ to b. On AC take $AS =$ to c. Draw RS.

From V draw $VW \parallel$ to RS, meeting AC at W.

Statement: SW is the fourth proportional required. Q.E.F.

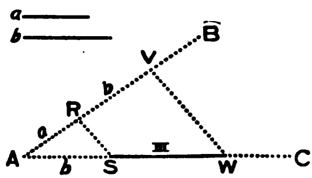
Proof: In $\triangle AVW$, RS is $\parallel$ to VW (Const.).

$\therefore\ a : b = c : SW$ (294).

Q.E.D.

PROPOSITION XLIII. PROBLEM

343. To find a third proportional to two given lines.



Given: (?).

Required: (?).

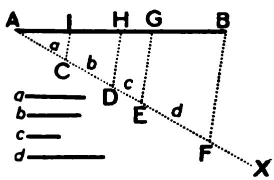
Construction:

Like that for 342.

Statement: (?). **Proof:** (?)

PROPOSITION XLIV. PROBLEM

344. To divide a given line into segments proportional to any number of given lines.



Given: AB; a, b, c, d.

Required: To divide AB into parts which shall be proportional to a, b, c, d.

Construction: Draw AX oblique to AB from A. On AX take $AC =$ to a, $CD =$ to b, $DE =$ to c, $EF =$ to d. Draw FB also through E, D, and C, lines $\parallel$ to FB, as EG, DH, and CI.

Statement: AI, IH, HG, GB are the required parts. Q.E.F.

Proof: $AI : a = IH : b = HG : c = GB : d$ (295).

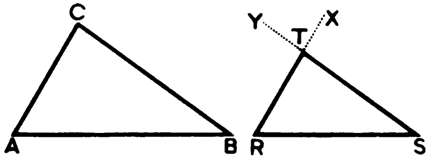
Q.E.D.

PROPOSITION XLV. PROBLEM

345. To construct a triangle similar to a given triangle and having a given side homologous to a side of the given triangle.

Given: $\triangle ABC$ and RS homologous to AB.

Required: To construct a $\triangle$ on RS similar to $\triangle ABC$.



Construction: At R construct $\angle SRX =$ to $\angle A$; at S construct $\angle RSY =$ to $\angle B$, the sides of these angles meeting at T.

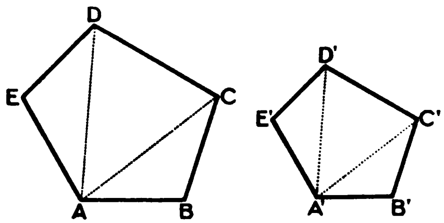
Statement: (?). **Proof:** (?). (303).

PROPOSITION XLVI. PROBLEM

346. To construct a polygon similar to a given polygon and having a given side homologous to a side of the given polygon.

Given: Polygon EB; line $A'B'$ homologous to AB.

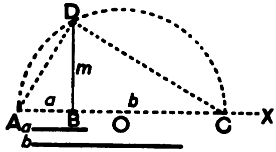
Required: To construct a polygon upon $A'B'$, similar to polygon EB.



Construction: From A draw diagonals AC and AD.
On $A'B'$ construct $\triangle A'B'C'$ similar to $\triangle ABC$ (by 345).
On $A'B'$ construct $\triangle A'C'D'$ similar to $\triangle ACD$. Etc.

Statement: (?). **Proof:** (?). (319).

Proposition XLVII. Problem



347. To find the mean proportional between two given lines.

Given: Lines a and b.

Required: To find the mean proportional between them.

Construction: On an indefinite line, AX, take $AB =$ to a and $BC =$ to b. Using O, the midpoint of AC, as center, and AO as radius, describe the semicircle, ADC. At B erect $BD \perp$ to AC, meeting the arc at D. Draw AD and CD.

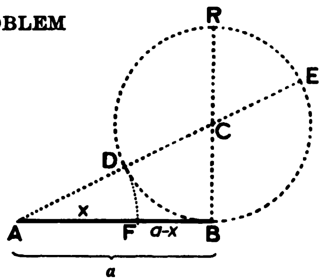
Statement: BD, or m, is the mean proportional required.

Q.E.F.

Proof: $a : m = m : b$. (?) Q.E.D.

348. A line is divided into **extreme and mean ratio** if one segment is a mean proportional between the whole line and the other segment; in other words, if a line is to one of its parts as that part is to the other part. (See 292.)

Proposition XLVIII. Problem



349. To divide a line into extreme and mean ratio.

Given: Line $AB = a$.

Required: To divide AB into extreme and mean ratio; that is, so that $AB : AF = AF : FB$.

Construction: At B erect BR, $\perp$ to AB and $=$ to AB. Using C, the midpoint of BR, as center, and CB as radius, describe a $\odot$. Draw AC meeting $\odot$ at D and E. On AB take $AF = AD$; let $AF = x$.

Statement: F divides AB so that $AB : AF = AF : FB$. Q.E.F.

Proof: AB is tangent to $\odot C$ (202).

$\therefore AE \cdot AD = \overline{AB}^2$ (324).

Now $AD = x$ (187) and $DE = a$ (190).

$\therefore AE = a + x$ (Ax. 4).

Substituting, $(a+x)x = a^2$ (Ax. 6).

Or $ax + x^2 = a^2$

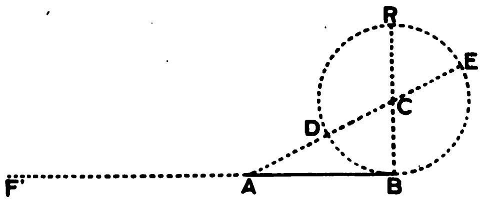
$\therefore x^2 = a^2 - ax = a(a - x)$

$\therefore a : x = x : a - x$ (281).

That is, $AB : AF = AF : FB$. Q.E.D.

PROPOSITION XLIX. PROBLEM.

350. To divide a line externally into extreme and mean ratio.



Given: (?).

Required: (?).

Construction: The same as in 349, except that AF' is taken on BA produced, = to AE.

Statement: $AB : AF' = AF' : BF'$. Q. E. F.

Proof: AB is tangent to $\odot C$ (202).

$\therefore AE : AB = AB : AD$ (325).

$\therefore AE + AB : AE = AB + AD : AB$ (284).

Now $AE + AB = BF'$ (Ax. 4).

Also $AB + AD = AE = AF'$ (Const.).

Substituting, $BF' : AF' = AF' : AB$ (Ax. 6).

That is, $AB : AF' = AF' : BF'$. Q.E.D.

351. The lengths of the several lines of 349 and 350 may be found by algebra, if the length of AB is known.

Thus if $AB = a$, we know in 349, $a : x = x : a - x$.

Hence $x^2 = a^2 - ax$. Solving this quadratic,

$x = AF = \frac{1}{2}a(\sqrt{5} - 1)$; also, $a - x = BF = \frac{1}{2}a(3 - \sqrt{5})$.

Likewise in 350, if $AB = a$, $AF' = y$, $a : y = y : a + y$.

Solving for y, $y = AF' = \frac{1}{2}a(\sqrt{5} + 1)$.

Also $a + y = BF' = \frac{1}{2}a(3 + \sqrt{5})$.

ORIGINAL CONSTRUCTIONS

It is required:

1. To construct a fourth proportional to lines that are exactly 3 in., 5 in., and 6 in. long. How long should this constructed line be?

2. To construct a mean proportional between lines that are exactly 4 in. and 9 in. How long should this constructed line be?

3. To construct a fourth proportional to three lines 5 in., 8 in., and 10 in. will this be the same length as a fourth proportional to 5 in., 10 in., and 8 in.? to 8 in., 10 in., and 5 in.? to 10 in., 5 in., and 8 in.?

4. To construct a third proportional to lines 3 in. and 6 in. long.

5. To produce a given line AB to point P, such that $AB : AP = 3 : 5$. [Divide AB into three equal parts, etc.]

6. To divide a line 8 in. long into two parts in the ratio of $5 : 7$. [Divide the given line into 12 equal parts.]

7. To solve Ex. 6 by constructing a triangle. [See 297.]

8. To divide one side of a triangle into segments proportional to the other two sides.

9. To divide one side of a triangle externally into segments proportional to the other sides.

10. To construct two straight lines having given their sum and ratio. [Consult Ex. 6.]

11. To construct two straight lines having given their difference and ratio. [Consult Ex. 5.]

12. To construct a triangle similar to a given triangle and having a given perimeter. [First, use 344.]

13. To construct a right triangle having given its perimeter and an acute angle. [Constr. a rt. $\triangle$ having the given acute $\angle$. Etc.]

14. To construct a triangle having given its perimeter and two angles. [Constr. a $\triangle$ having the two given $\angle$s. Etc.]

15. To construct a triangle similar to a given triangle and having a given altitude.

16. To construct a rectangle similar to a given rectangle and having a given base.

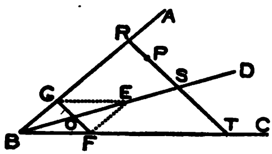
17. To construct a rectangle similar to a given rectangle and having a given perimeter.

18. To construct a parallelogram similar to a given parallelogram and having a given base.

19. To construct a parallelogram similar to a given parallelogram and having a given perimeter.

20. To construct a parallelogram similar to a given parallelogram, and having a given altitude.

21. To draw through a given point another line, which is terminated by the outer two of three lines meeting in a point and bisected by the inner one.



Construction: From E on BD draw ‖s. Etc. Through P draw $RT \parallel$ to GF.

Statement: $RS = ST$.

22. To inscribe in a given circle a triangle similar to a given triangle.

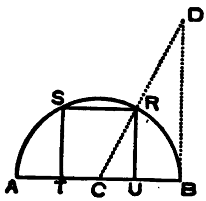
Construction: Circumscribe a ⊙ about the given △; draw radii to the vertices; at center of given ⊙ construct 3 ∡ = to the other central angles.

23. To circumscribe about a given circle a triangle similar to a given triangle.

Construction: First, inscribe a △ similar to the given △.

24. To construct a right triangle, having given one leg and its projection upon the hypotenuse.

25. To inscribe a square in a given semicircle.



Construction: At B erect $BD \perp$ to AB and $=$ to AB; draw DC, meeting ⊙ at R; draw $RU \parallel$ to BD. Etc.

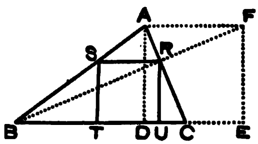
Statement: (?).

Proof: $RSTU$ is a rectangle (?). △ CRU is similar to △ CDB (?). $\therefore CU : CB = UR : BD$ (?). But $CB = \frac{1}{2} BD$ (?). $\therefore CU = \frac{1}{2} UR$ (?). Etc.

26. To inscribe in a given semicircle a rectangle similar to a given rectangle.

Construction: From the midpoint of the base draw line to one of the opposite vertices. At given center construct an ∠ = to the ∠ at the midpoint. Proceed as in Ex. 25.

27. To inscribe a square in a given triangle.



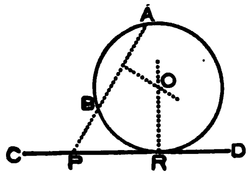
Construction: Draw altitude AD; construct the square $ADEF$ upon AD as a side; draw BF meeting AC at R.

Draw $RU \parallel$ to AD; $RS \parallel$ to BC. Etc.

28. To inscribe in a given triangle a rectangle similar to a given rectangle.

Construction: Draw the altitude. On this construct a rectangle similar to the given rectangle.

Proceed as in Ex. 27.



29. To construct a circle which shall pass through two given points and touch a given line.

Given: Points A and B; line CD.

Construction: Draw line AB meeting CD at P. Construct a mean proportional between PA and PB (by 347). On PD take $PR =$ to this mean. Erect $OR \perp$ to CD at R, meeting $\perp$ bisector of AB at O. Use O as center, etc.

30. To construct a line $=$ to $\sqrt{2}$ in. [Diag. of square the side of which is 1 in.]

31. To construct a line $=$ to $\sqrt{5}$ in.

[Hyp. of a rt. $\triangle$, whose legs are 1 in. and 2 in. respectively.]

32. To divide a line into segments in the ratio of $1 : \sqrt{2}$.

33. To divide a line into segments in the ratio of $1 : \sqrt{5}$.

34. To construct a line x, if $x = \frac{ab}{c}$, and a, b, c are lengths of three given lines. [That is, to construct x, if $c : a = b : x$ (281).]

35. To construct a line x, if $x = \frac{ab}{3c}$. [$3c : a = b : x$.]

36. To construct a line x, if $x = \sqrt{ab}$. [$a : x = x : b$.]

37. To construct a line x, if $x = \frac{a^2}{c}$.

38. To construct a line x, if $x = \sqrt{a^2 - b^2}$. [$a + b : x = x : a - b$.]

39. To construct a line x, if $x = \frac{2a^2}{c}$.

40. To construct a line y, if $ay = \frac{2}{3} b^2$.

41. To construct a line $=$ to $\sqrt{10}$ in.

42. To construct a line $=$ to $2\sqrt{6}$ in.

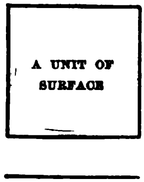
43. To construct a line $=$ to $\sqrt{a^2 + b^2}$, if a and b are given lines.

BOOK IV

AREAS OF POLYGONS

THEOREMS AND DEMONSTRATIONS

352. The **unit of surface** is a square each side of which is a unit of length.



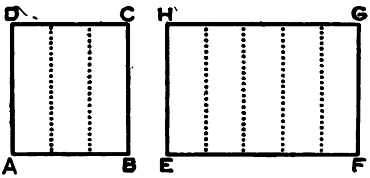
UNIT OF LENGTH

353. The **area of a surface** is the *number* of units of surface it contains. The area of a surface is the ratio of that surface to the unit of surface.

NOTE. It is often convenient to speak of "triangle," "rectangle," etc., when one really means "the area of a triangle," or "the area of a rectangle," etc.

PROPOSITION I. THEOREM

354. If two rectangles have equal altitudes, they are to each other as their bases.



Given: Rectangles AC and EG having equal altitudes, with bases AB and EF.

To Prove:

$AC : EG = AB : EF$.

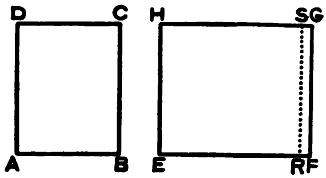
Proof: I. If AB and EF are **commensurable.**

There is a common unit of measure of AB and EF (225). Suppose this unit is contained 3 times in AB and 5 times in EF. Hence $\quad AB : EF = 3 : 5 \quad$ (Ax. 3).

Draw lines through these points of division $\perp$ to the bases. These divide rectangle AC into three parts and EG into 5 parts, and all of these eight parts are equal (134).

Hence $\quad AC : EG = 3 : 5 \quad$ (Ax. 3).

$\therefore AC : EG = AB : EF$ (Ax. 1). Q.E.D.



II. If AB and EF are **incommensurable.**

There does not exist a common unit (225).

Divide AB into several equal parts. Apply one of these as a unit of measure to EF.

There is a remainder, RF (Hyp.).

Draw $RS \perp$ to EF. Now $\frac{AC}{ES} = \frac{AB}{ER}$ (Case I).

Indefinitely increase the number of equal parts of AB; that is, indefinitely decrease each part, or the unit or divisor. Then the remainder, RF, is indefinitely decreased.

That is, RF approaches zero as a limit,

Also $RFGS$ approaches zero as a limit.

Hence ER approaches EF as a limit,

Also ES approaches EG as a limit.

Therefore $\frac{AC}{ES}$ approaches $\frac{AC}{EG}$ as a limit,

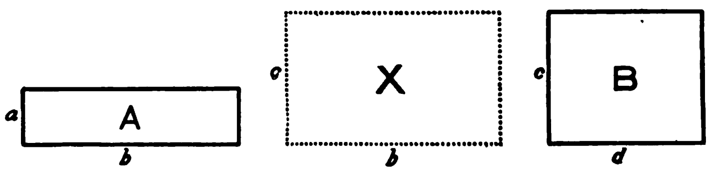
Also $\frac{AB}{ER}$ approaches $\frac{AB}{EF}$ as a limit.

$\therefore \frac{AC}{EG} = \frac{AB}{EF}$ (229). Q.E.D.

355. Corollary. **Two rectangles having equal bases are to each other as their altitudes.** (Explain.)

Proposition II. Theorem

356. Any two rectangles are to each other as the products of their bases by their altitudes.



Given: Rectangles A and B the altitudes of which are a and c, and the bases b and d, respectively.

To Prove: $A : B = a \cdot b : c \cdot d$.

Proof: Construct a third rectangle X, whose base is b and whose altitude is c.

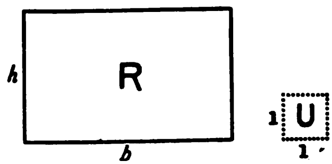
Then $\frac{A}{X} = \frac{a}{c}$ (355).

Also $\frac{X}{B} = \frac{b}{d}$ (354).

Multiplying, $\frac{A}{B} = \frac{a \cdot b}{c \cdot d}$ (Ax. 3). Q.E.D.

Proposition III. Theorem

357. The area of a rectangle is equal to the product of its base by its altitude.



Given: Rectangle R, with base b and altitude h.

To Prove: Area of $R = b \cdot h$.

Proof: Draw a square U, each side of which is a unit of length. This square is a unit of surface (352).

Now $\frac{R}{U} = \frac{b \cdot h}{1 \cdot 1} = b \cdot h$ (356).

But $\frac{R}{U} =$ the area of R (353).

$\therefore$ the area of $R = b \cdot h$ (Ax. 1).

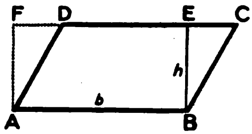
Q.E.D.

358. Corollary. The area of a square is equal to the square of its side. (357.)

Ex. I have enough material to build 1000 yards of fence. If I put this around a square field, how many square yards will the field contain? If I put it around a rectangular field that is four times as long as it is wide, how many square yards will the field contain?

Proposition IV. Theorem

359. The area of a parallelogram is equal to the product of its base by its altitude.



Given: ▱ $ABCD$, with base b and altitude h.

To Prove: Area of $ABCD = b \cdot h$.

Proof: From A and B, the extremities of the base, draw ⊥s to the upper base meeting it in F and E respectively.

In rt. △s ADF and BCE,

$AF = BE$ (124).

$AD = BC$ (124).

∴ △ ADF is congruent to △ BCE (84).

Now from the whole figure subtract △ ADF and the parallelogram $ABCD$ remains. And from the whole figure subtract △ BCE and rectangle $ABEF$ remains.

∴ ▱ $ABCD$ = rectangle $ABEF$ (Ax. 2).

But rectangle $ABEF = b \cdot h$ (357).

∴ ▱ $ABCD = b \cdot h$ (Ax. 1).

Q.E.D.

360. Corollary. **All parallelograms having equal bases and equal altitudes are equal in area.**

361. Corollary. **Two parallelograms having equal altitudes are to each other as their bases.**

Proof: $P = b \cdot h$ and $P' = b' \cdot h$ (359).

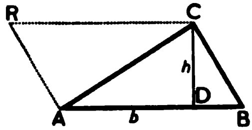
Dividing, $\frac{P}{P'} = \frac{b \cdot h}{b' \cdot h} = \frac{b}{b'}$ (Ax. 3).

Q.E.D.

362. Corollary. **Two parallelograms having equal bases are to each other as their altitudes. Proof:** (?).

363. Corollary. **Any two parallelograms are to each other as the products of their bases by their altitudes. Proof:** (?).

PROPOSITION V. THEOREM



364. The area of a triangle is equal to half the product of its base by its altitude.

Given: $\triangle ABC$, with base b and altitude h.

To Prove: Area of $\triangle ABC = \frac{1}{2} b \cdot h$.

Proof: Through A draw $AR \parallel$ to BC and through C draw $CR \parallel$ to AB, meeting AR at R.

Now $ABCR$ is a $\square$ (120).

The area of $\square ABCR = b \cdot h$ (359).

Dividing by 2, $\frac{1}{2} \square ABCR = \frac{1}{2} b \cdot h$ (Ax. 3).

Also $\frac{1}{2} \square ABCR = \triangle ABC$ (126).

$\therefore$ the area of $\triangle ABC = \frac{1}{2} b \cdot h$ (Ax. 1). Q.E.D.

365. COROLLARY. **A triangle having the same base and altitude as a parallelogram equals half the parallelogram.**

366. COROLLARY. **All triangles having equal bases and equal altitudes are equal in area.**

367. COROLLARY. **All triangles having the same base and whose vertices are in a line parallel to the base are equal.**

368. COROLLARY. **Two triangles having equal altitudes are to each other as their bases.**

Proof: $\triangle T = \frac{1}{2} b \cdot h$; and $\triangle T' = \frac{1}{2} b'h$ (364).

Dividing, $\frac{\triangle T}{\triangle T'} = \frac{\frac{1}{2} bh}{\frac{1}{2} b'h} = \frac{b}{b'}$ (Ax. 3).

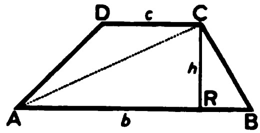
369. COROLLARY. **Two triangles having equal bases are to each other as their altitudes. Proof:** (?).

370. COROLLARY. **Any two triangles are to each other as the products of their bases by their altitudes. Proof:** (?).

371. COROLLARY. **The area of a right triangle is equal to half the product of the legs. Proof:** (?).

Proposition VI. Theorem

372. The area of a trapezoid is equal to half the product of the altitude by the sum of the bases.



Given: Trapezoid $ABCD$, with altitude h and bases b and c.

To Prove: Area $= \frac{1}{2} h \cdot (b + c)$.

Proof: Draw diagonal AC. The $\triangle$s ABC and ADC have the same altitude, h, and their bases are b and c, respectively.

Now $\triangle ABC = \frac{1}{2} b \cdot h$ (364).

Also $\triangle ADC = \frac{1}{2} c \cdot h$ (?).

Adding, $\triangle ABC + \triangle ADC = \frac{1}{2} b \cdot h + \frac{1}{2} c \cdot h$ (Ax. 2).

That is, trapezoid $ABCD = \frac{1}{2} h \cdot (b + c)$ (Ax. 6). Q.E.D.

373. Corollary. The area of a trapezoid is equal to the product of the altitude by the median.

Proof: Area $ABCD = \frac{1}{2} h \cdot (b + c) = h \cdot \frac{1}{2}(b + c)$ (372).

But $\frac{1}{2}(b + c) =$ median (138).

Hence area of trapezoid $ABCD = h \cdot m$ (Ax. 6). Q.E.D.

Ex. 1. If one parallelogram has half the base and the same altitude as another, the area of the first is half the area of the second.

Ex. 2. If one parallelogram has half the base and half the altitude of another, its area is one fourth the area of the second.

Ex. 3. State and prove two analogous theorems about triangles.

Ex. 4. If a triangle has half the base and half the altitude of a parallelogram, the triangle is one eighth of the parallelogram.

Ex. 5. The area of a rhombus equals half the product of its diagonals.

Ex. 6. The diagonals of a parallelogram divide it into four triangles of equal areas.

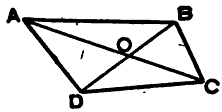
Ex. 7. The diagonals of a trapezoid divide it into four triangles, two of which are similar and the other two have equal areas.

Ex. 8. If a parallelogram has half the base and half the altitude of a triangle, its area is half the area of the triangle.

Ex. 9. The line joining the midpoints of two sides of a triangle forms a triangle whose area is one fourth the area of the original triangle.

Ex. 10. The line joining the midpoints of two adjacent sides of a parallelogram cuts off a triangle whose area is one eighth of the area of the parallelogram.

Ex. 11. If one diagonal of a quadrilateral bisects the other, it also divides the quadrilateral into two triangles having equal areas.



To Prove: $\triangle ABC = \triangle ADC$.

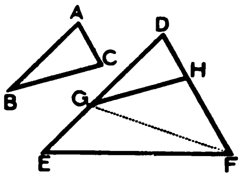
Ex. 12. Either diagonal of a trapezoid divides the figure into two triangles the ratio of which is equal to the ratio of the bases of the trapezoid. Prove in two ways.

Ex. 13. If, in triangle ABC, D and E are the midpoints of sides AB and AC respectively, $\triangle BCD = \triangle BEC$.

Ex. 14. If the diagonals of quadrilateral $ABCD$ meet at E, and $\triangle ABE$ is equal in area to $\triangle CDE$, the sides AD and BC are parallel.

PROPOSITION VII. THEOREM

374. If two triangles have an angle of one equal to an angle of the other, they are to each other as the products of the sides including the equal angles.



Given: ◬ ABC and DEF, $\angle A = \angle D$.

To Prove: $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$.

Proof: Superpose $\triangle ABC$ upon $\triangle DEF$ so that the equal ∡ coincide and BC takes the position denoted by GH. Draw GF.

Now ◬ DGH and DGF have the same altitude (a ⊥ from G to DF), and ◬ DGF and DEF have the same altitude (a ⊥ from F to DE).

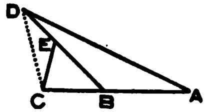
$$\therefore \frac{\triangle DGH}{\triangle DGF} = \frac{DH}{DF} \quad \text{and} \quad \frac{\triangle DGF}{\triangle DEF} = \frac{DG}{DE} \qquad (368).$$

Multiplying,
$$\frac{\triangle DGH}{\triangle DEF} = \frac{DG \cdot DH}{DE \cdot DF} \qquad (\text{Ax. } 3).$$

That is,
$$\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot AC}{DE \cdot DF} \qquad (\text{Ax. } 6).$$

Q.E.D.

Ex. 1. If two triangles have an angle of one the supplement of an angle of the other, the triangles are to each other as the products of the sides including these angles.

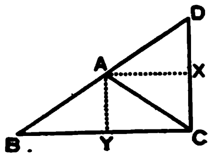


Ex. 2. If two triangles of equal area have an angle of one equal to an angle of the other, the sides including these angles are reciprocally proportional.

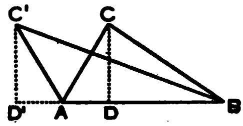
Ex. 3. Any two sides of a triangle are reciprocally proportional to the altitudes upon them.

Ex. 4. In triangles of equal area the bases and the altitudes upon them are reciprocally proportional.

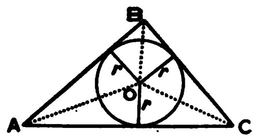
Ex. 5. If two isosceles triangles have the legs of one equal to the legs of the other, and the vertex angle of the one the supplement of the vertex angle of the other, the triangles have equal areas.



Ex. 6. Two triangles are equal in area if they have two sides of one equal to two sides of the other and the included angles supplementary.



Ex. 7. The area of a triangle is equal to half the perimeter of the triangle multiplied by the radius of the inscribed circle.



Ex. 8. The area of a polygon circumscribed about a circle is equal to half the product of the perimeter of the polygon by the radius of the circle.

Ex. 9. The line joining the midpoints of the bases of a trapezoid bisects the area of the trapezoid.

Ex. 10. Any line drawn through the midpoint of a diagonal of a parallelogram, intersecting two sides, bisects the area of the parallelogram.

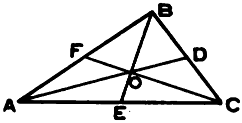
Ex. 11. The lines joining (in order) the midpoints of the sides of any quadrilateral form a parallelogram whose area is half the area of the quadrilateral.

Ex. 12. If any point within a parallelogram is joined to the four vertices, the sum of one pair of opposite triangles is equal to the sum of the other pair; that is, to half the parallelogram.

Ex. 13. Is a triangle bisected by an altitude? by the bisector of an angle? by a median? by the perpendicular bisector of a side? Give reasons.

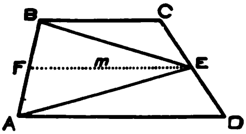
Ex. 14. If the three medians of a triangle are drawn, there are six pairs of triangles formed, one of each pair being double the other.

For instance, $\triangle AOB = 2 \triangle AOE$; etc.



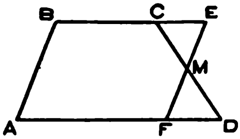
Ex. 15. If the midpoints of two sides of a triangle are joined to any point in the base, the quadrilateral formed is half the original triangle.

Ex. 16. If lines are drawn from the midpoint of one leg of a trapezoid to the ends of the other leg, the middle triangle thus formed is equivalent to half the trapezoid.



Ex. 17. The area of a trapezoid is equal to the product of one of the non-parallel sides, by the perpendicular upon it from the midpoint of the other.

Ex. 18. If through the midpoint of one of the non-parallel sides of a trapezoid a line is drawn parallel to the other side, the parallelogram formed is equivalent to the trapezoid.



Ex. 19. The sum of the three perpendiculars drawn to the three sides of an equilateral triangle from any point within is constant (being equal to the altitude of the triangle).

Proof: Join the point to the vertices. Set the sum of the areas of the three inner $\triangle$ equal to the area of the whole $\triangle$. Etc.

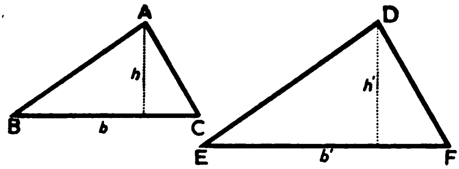
Historical Note. Sir Isaac Newton was born on Dec. 25, 1642, at Grantham, England. At an early age he exhibited a fondness and aptitude for mechanical contrivances, — windmills, water-clocks, kites and dials.

NEWTON

Later in his career he studied Descartes' geometry and was inspired with a love for all the mathematical studies. In the years 1665 and 1666 he made many important mathematical discoveries. He invented improvements for both the telescope and microscope, and discovered the existence of the spectrum. The study of Kepler's laws of motion resulted in Newon's discovery of the law of gravitation, which he discussed with such contemporaries as Sir Christopher Wren, Hooke, and the astronomer Halley. He also discovered the binomial theorem, which was inscribed on his tomb when he died in 1727. He has always been honored as the greatest mathematician of all time.

Proposition VIII. Theorem

375. Two similar triangles are to each other as the squares of any two homologous sides.



Given: Similar $\triangle$ *ABC* and *DEF*.

To Prove: $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\overline{AB}^2}{\overline{DE}^2} = \dfrac{\overline{AC}^2}{\overline{DF}^2} = \dfrac{\overline{BC}^2}{\overline{EF}^2}$.

Proof: Denote a pair of homologous altitudes by h and h', and the corresponding bases by b and b'.

Now $$\frac{\triangle ABC}{\triangle DEF} = \frac{b \cdot h}{b' \cdot h'} = \frac{b}{b'} \cdot \frac{h}{h'} \qquad (370).$$

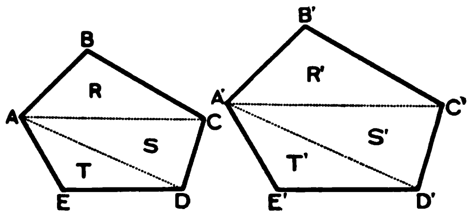
But $$\frac{h}{h'} = \frac{b}{b'} \qquad (311).$$

Substituting, $$\frac{\triangle ABC}{\triangle DEF} = \frac{b}{b'} \cdot \frac{b}{b'} = \frac{b^2}{b'^2} \qquad (\text{Ax. } 6).$$

That is, $$\frac{\triangle ABC}{\triangle DEF} = \frac{\overline{BC}^2}{\overline{EF}^2} \text{ or } = \frac{\overline{AB}^2}{\overline{DE}^2} \text{ or } = \frac{\overline{AC}^2}{\overline{DF}^2}. \qquad \text{Q.E.D.}$$

Proposition IX. Theorem

376. Two similar polygons are to each other as the squares of any two homologous sides and as the squares of their perimeters.



Given: Similar polygons *ABCDE* and $A'B'C'D'E'$, with perimeters P and P' respectively.

To Prove: $\dfrac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \text{etc., and} = \dfrac{P^2}{P'^2}$.

Proof: Draw from homologous vertices, A and A', all the pairs of homologous diagonals, dividing the polygons into △s.

These △s are similar in pairs. (318).

$$\frac{\triangle R}{\triangle R'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad (375).$$

$$\frac{\triangle S}{\triangle S'} = \frac{\overline{CD}^2}{\overline{C'D'}^2} \qquad (?).$$

$$\frac{\triangle T}{\triangle T'} = \frac{\overline{DE}^2}{\overline{D'E'}^2} \qquad (?).$$

But $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} \qquad (313).$$

$$\therefore \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \frac{\overline{DE}^2}{\overline{D'E'}^2} \qquad (287).$$

$$\therefore \frac{\triangle R}{\triangle R'} = \frac{\triangle S}{\triangle S'} = \frac{\triangle T}{\triangle T'} \qquad (\text{Ax. } 1).$$

$$\therefore \frac{\triangle R + \triangle S + \triangle T}{\triangle R' + \triangle S' + \triangle T'} = \frac{\triangle R}{\triangle R'} \qquad (291).$$

Substituting, $$\frac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \frac{\triangle R}{\triangle R'} \qquad (\text{Ax. } 6).$$

But $$\frac{\triangle R}{\triangle R'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad (\text{Above}).$$

$$\therefore \frac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad (\text{Ax. } 6).$$

Q.E.D.

Also $$P : P' = AB : A'B' = \text{etc.} \qquad (317).$$

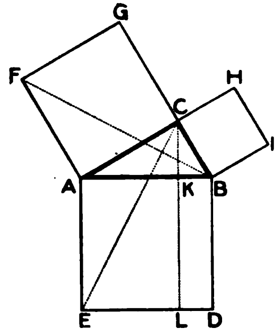
$$\therefore P^2 : P'^2 = \overline{AB}^2 : \overline{A'B'}^2 = \text{etc.} \qquad (287).$$

$$\therefore \frac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \frac{P^2}{P'^2}. \qquad (\text{Ax. } 1).$$

Q.E.D.

Proposition X. Theorem

377. The square described upon the hypotenuse of a right triangle is equal in area to the sum of the squares described upon the legs.



Given: (?).

To Prove: (?).

Proof: Draw $CL \perp$ to AB, meeting AB at K and ED at L. Draw BF and CE.

Now	$\angle$s ACB, ACG, and BCH are all rt. $\angle$s.	(Hyp.)
Hence	ACH and BCG are straight lines.	(45.)
Also	$AELK$ and $BDLK$ are rectangles.	(Def.)
In	$\triangle$s ABF and ACE, $AB = AE$, $AF = AC$	(122).

$$\angle BAF = \angle CAE$$

(Each is composed of a rt. $\angle$ plus $\angle CAB$).

$$\therefore \triangle ABF \cong \triangle ACE \quad (52).$$

Also $\triangle ABF$ and square AG have the same base, AF, and the same altitude, AC.

$$\therefore \text{square } AG = 2 \cdot \triangle ABF \quad (365).$$

Similarly, rectangle $AKLE = 2 \cdot \triangle ACE$ (365).

$$\therefore \text{rectangle } AKLE = \text{square } AG \quad (\text{Ax. } 1).$$

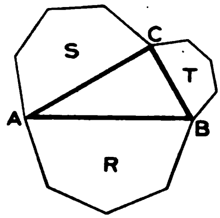
By drawing AI and CD, it may be proved in the same manner that rectangle $BDLK$ = square BH.

By adding, square AD = square AG + square BH (Ax. 2).

Q.E.D.

378. Corollary. The square described upon one of the legs of a right triangle is equal in area to the square described upon the hypotenuse minus the square described upon the other leg.

Proposition XI. Theorem



379. If the three sides of a right triangle are the homologous sides of three similar polygons, the polygon described upon the hypotenuse is equal in area to the sum of the two polygons described upon the legs.

Given: (?). **To Prove:** (?).

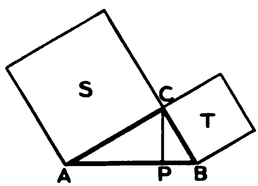
Proof: $\frac{S}{R}=\frac{\overline{AC}^2}{\overline{AB}^2}$ (376).

And $\frac{T}{R}=\frac{\overline{BC}^2}{\overline{AB}^2}$ (?).

Adding, $\frac{S+T}{R}=\frac{\overline{AC}^2+\overline{BC}^2}{\overline{AB}^2}=\frac{\overline{AB}^2}{\overline{AB}^2}=1.$ (Ax. 2; 334.)

Clearing of fractions, $R=S+T$ (Ax. 3). Q.E.D.

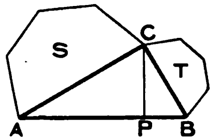
380. Corollary. If the three sides of a right triangle are the homologous sides of three similar polygons, the polygon described upon one of the legs is equal in area to the polygon described upon the hypotenuse minus the polygon described upon the other leg.



381. Corollary. The two squares described upon the legs of a right triangle are to each other as the projections of the legs upon the hypotenuse.

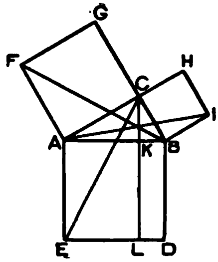
Proof: Square $S=\overline{AC}^2$ Square $T=\overline{BC}^2$ (?).

$\therefore \frac{\text{Square } S}{\text{Square } T}=\frac{\overline{AC}^2}{\overline{BC}^2}=\frac{AB\cdot AP}{AB\cdot BP}=\frac{AP}{BP}$ (Ax. 3; 333).



382. Corollary. If two similar polygons are described upon the legs of a right triangle as homologous sides, they are to each other as the projections of the legs upon the hypotenuse.

Proof: $$\frac{\text{Polygon } S}{\text{Polygon } T} = \frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \cdot AP}{AB \cdot BP} = \frac{AP}{BP} \quad (376;\ 333).$$



Ex. 1. In the figure of 377, prove that:

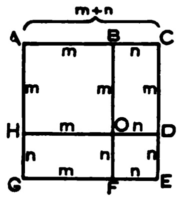
(*i*) Points I, C, and F are in a straight line.

(*ii*) CE and BF are perpendicular.

(*iii*) AG and BH are parallel.

(*iv*) $\triangle AEF = \triangle CGH = \triangle BDI = \triangle ABC$.

Ex. 2. The sum of the squares described upon the four segments of two perpendicular chords in a circle is equal to the square described upon the diameter. (Fig. is on page 185.)

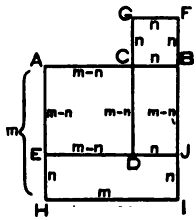


Ex. 3. The square described upon the sum of two lines is equal to the sum of the squares described upon the two lines, plus twice the rectangle of these lines.

To Prove: Square $AE = m^2 + n^2 + 2\,mn$.

Ex. 4. The square described upon the difference of two lines is equal to the sum of the squares described upon the two lines minus twice the rectangle of these lines.

To Prove: Square $AD = m^2 + n^2 - 2\,mn$.



Ex. 5. A and B are the extremities of a diameter of a circle; C and D are the points of intersection of any third tangent to this circle with the tangents at A and B respectively. Prove that the area of $ABDC$ is equal to $\frac{1}{2}\,AB \cdot CD$.

Ex. 6. If the four points midway between the center and vertices of a parallelogram are joined in order, a parallelogram is formed similar to the original parallelogram; its perimeter is half of the perimeter of the original figure; and its area is one quarter of the area of the original figure.

Ex. 7. If two triangles of equal area have the same base and lie on opposite sides of it, the line joining their vertices is bisected by the line of the base.

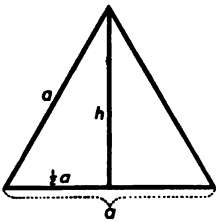
Ex. 8. What part of a right triangle is the quadrilateral which is cut from the triangle by a line joining the midpoints of the legs?

Ex. 9. From M, a vertex of parallelogram $LMNO$, a line MPX is drawn meeting NO at P and LO produced, at X. LP and NX are also drawn. Prove that triangles LOP and XNP are equal in area.

FORMULAS

Proposition XII. Problem

383. To derive the formulas for the altitude and the area of an equilateral triangle, in terms of its side.



Solution: Let each side $= a$, and altitude $= h$. $h^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$ (335).

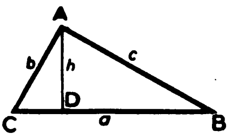
I. $\therefore h = \frac{a}{2}\sqrt{3}.$

$$\text{Area} = \frac{1}{2}a \cdot h = \frac{1}{2}a \cdot \frac{a}{2}\sqrt{3} \quad (364).$$

II. $\therefore \textbf{Area} = \frac{a^2\sqrt{3}}{4}.$

Proposition XIII. Problem

384. To derive a formula for the area of a triangle in terms of its sides.



Given: $\triangle ABC$, having sides a, b, c.

Required: To derive a formula for its area, containing only a, b, and c.

Solution: Draw altitude AD.

Now $$CD = {}_bp_a = \frac{a^2 + b^2 - c^2}{2a} \quad (339).$$

Also $$\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2 \quad (335).$$

$$\therefore h^2 = b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2 \quad (\text{Ax. } 6).$$

Hence $h^2 = \left\{b + \dfrac{a^2+b^2-c^2}{2a}\right\}\left\{b - \dfrac{a^2+b^2-c^2}{2a}\right\}$ (by factoring).

Also $h^2 = \dfrac{2ab + a^2 + b^2 - c^2}{2a} \cdot \dfrac{2ab - a^2 - b^2 + c^2}{2a}.$

$$\therefore h = \sqrt{\frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}}.$$

Let $a + b + c = 2s$.

Then it is evident that $a + b - c = 2(s - c)$
and $a - b + c = 2(s - b)$ and $-a + b + c = 2(s - a)$.

Substituting above, $h = \sqrt{\dfrac{2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)}{4a^2}}$

$$= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

Now area $\triangle = \dfrac{1}{2} a \cdot h = \dfrac{a}{2} \cdot \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$

$$\therefore \textbf{Area of } \triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

Exercise. Find the area of a triangle whose sides are 17, 25, 28.
Here, $a = 17$, $b = 25$, $c = 28$, $s = 35$, $s - a = 18$, $s - b = 10$, $s - c = 7$.
Area $= \sqrt{35 \cdot 18 \cdot 10 \cdot 7} = \sqrt{7^2 \cdot 5^2 \cdot 2^2 \cdot 3^2} = 210$.

Ex. 1. Find the area of the triangle whose sides are 7, 10, 11.

Ex. 2. Find the area of the triangle whose sides are 8, 15, 17.

Ex. 3. Find the area of the equilateral triangle whose side is 8.

Ex. 4. Find the side of the equilateral triangle whose area is $121\sqrt{3}$.

Ex. 5. Find the area of the equilateral triangle whose altitude is 10.

Proposition XIV. Problem

385. To derive formulas for the altitudes of a triangle in terms of the three sides.

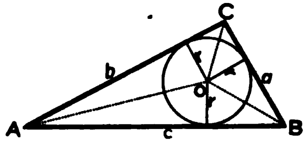
Solution: Area $= \frac{1}{2} a h_a = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore h_a = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}a}.$$

Similarly, $h_b = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}b}$; $h_c = \dfrac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}c}$.

PROPOSITION XV. PROBLEM

386. To derive the formula for the radius of the circle inscribed in a triangle, in terms of the sides of the triangle.



Solution:

$$\left.\begin{aligned}\text{Area of } \triangle AOB &= \tfrac{1}{2}c\cdot r\\ \text{area of } \triangle AOC &= \tfrac{1}{2}b\cdot r\\ \text{area of } \triangle BOC &= \tfrac{1}{2}a\cdot r\end{aligned}\right\} \qquad (?).$$

Adding, $\quad$ area of $\triangle ABC = \frac{1}{2}(a+b+c)r = sr.$

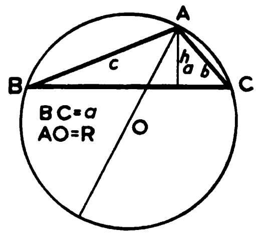
[Because $\quad \frac{1}{2}(a+b+c) = s.$]

Hence $$r = \frac{\text{area of } \triangle ABC}{s}.$$

$$\therefore r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

PROPOSITION XVI. PROBLEM

387. To derive the formula for the radius of the circle circumscribed about a triangle, in terms of the sides of the triangle.



Solution:

$$2R\cdot h_a = b\cdot c \qquad (328).$$

$$\therefore R = \frac{b\cdot c}{2h_a}.$$

$$h_a = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}a} \qquad (385).$$

$$\therefore R = \frac{a\cdot b\cdot c}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Ex. 1. Find the radius of the circle inscribed in, and the radius of the circle circumscribed about, the triangle whose sides are 17, 25, 28.

Ex. 2. Find for triangle whose sides are 11, 14, 17, the radii of the inscribed and circumscribed circles.

ORIGINAL EXERCISES (NUMERICAL)

1. The base of a parallelogram is 2 ft. 6 in. and its altitude is 1 ft. 4 in. Find the area. Find the side of a square of equal area.

2. The area of a rectangle is 540 sq. m. and its altitude is 15 m. Find its base and its diagonal.

3. The base of a rectangle is 3 ft. 4 in. and its diagonal is 3 ft. 5 in. Find its area.

4. The bases of a trapezoid are 2 ft. 1 in., and 3 ft. 4 in., and the altitude is 1 ft. 2 in. Find the area.

5. The area of a trapezoid is 736 sq. in. and its bases are 3 ft. and 4 ft. 8 in. Find the altitude.

6. The area of a certain triangle whose base is 40 rd. is 3.2 A. Find the area of a similar triangle whose base is 10 rd. Find the altitudes of these triangles.

7. The base of a certain triangle is 20 cm. Find the base of a similar triangle four times as large; of one five times as large; twice as large; half as large; one ninth as large.

8. The altitude of a certain triangle is 12 and its area is 100. Find the altitude of a similar triangle three times as large. Find the base of a similar triangle seven times as large. Find the altitude and the base of a similar triangle one third as large.

9. The area of a polygon is 216 sq. m. and its shortest side is 8 m. Find the area of a similar polygon whose shortest side is 10 m. Find the shortest side of a similar polygon four times as large; one tenth as large.

10. If the longest side of a polygon whose area is 567 is 14, what is the area of a similar polygon whose longest side is 12? of another whose longest side is 21?

11. Find the area of an equilateral triangle whose sides are each 6 in.; of another whose sides are each $10\sqrt{3}$ ft.

12. Find the area of an equilateral triangle whose altitude is 4 in.; of another whose altitude is 18 dm.

13. The area of an equilateral triangle is $64\sqrt{3}$. Find its side and its altitude.

14. The area of an equilateral triangle is 90 sq. m. Find its altitude.

15. Find the side of an equilateral triangle whose area is equal to a square whose side is 15 ft.

16. The equal sides of an isosceles triangle are each 17 in. and the base is 16 in. Find the area.

17. Find the area of an isosceles right triangle whose hypotenuse is 2 ft. 6 in.

18. Find the area of a square whose diagonal is 20 m.

19. There are two equilateral triangles whose sides are 33 and 56 respectively. Find the side of the equilateral triangle equal to their sum. Find the side of the equilateral triangle equal to their difference.

20. There are two similar polygons two of whose homologous sides are 24 and 70. Find the side of a third similar polygon equal to their sum; the side of a similar polygon equal to their difference.

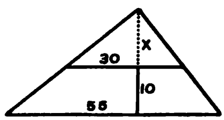
21. What is the area of the right triangle whose hypotenuse is 29 cm. and whose short leg is 20 cm.?

22. The base of a triangle is three times the base of a triangle equal in area. What is the ratio of their altitudes?

23. The bases of a trapezoid are 56 ft. and 44 ft. and the non-parallel sides are each 10 ft. Find its area. Also find the diagonal of a square of equal area.

24. The base of a triangle is 80 m., and its altitude is 8 m. Find the area of the triangle cut off by a line parallel to the base and at a distance of 3 m. from it. Find the area of another triangle, cut off by a line parallel to the base and 6 m. from it.

25. The bases of a trapezoid are 30 and 55, and its altitude is 10. If the non-parallel sides are produced till they meet, find the area of the less triangle formed.



[The ⧍s are similar. $\therefore 30:55 = x : x+10$. Etc.]

26. The diagonals of a rhombus are 2 ft. and 70 in. Find the area; the perimeter; the altitude.

27. The altitude (h) of a triangle is increased by n and the base (b) is diminished by x so that the area remains unchanged. Find x.

28. The projections of the legs of a right triangle upon the hypotenuse are 8 and 18. Find the area of the triangle.

29. In triangle ABC, AB is 5, BC is 8, and AB is produced to P, making $BP=6$. BC is produced (through B) to Q and PQ is drawn so that triangle BPQ is equal in area to triangle ABC. Find the length of BQ.

30. The angle C of triangle ABC is right; $AC = 5$; $BC = 12$. BA is produced through A, to D making $AD = 4$; CA is produced through A, to E so that triangle AED is equal in area to triangle ABC. Find AE.

31. Find the area of a square inscribed in a circle whose radius is 6.

32. Find the side of an equilateral triangle whose area is $25\sqrt{3}$.

33. Two sides of a triangle are 12 and 18. What is the ratio of the two triangles formed by the bisector of the angle between these sides?

34. The perimeter of a rectangle is 28 m. and one side is 5 m. Find the area.

35. The perimeter of a polygon is 5 ft. and the radius of the inscribed circle is 5 in. Find the area of the polygon.

In the following triangles, find the area, the three altitudes, the radius of the inscribed circle, the radius of circumscribed circle:

36. $a = 13$, $b = 14$, $c = 15$.

37. $a = 15$, $b = 41$, $c = 52$.

38. 20, 37, 51.

39. 140, 143, 157.

40. The sides of a triangle are 15, 41, 52. Find the areas of the two triangles into which this triangle is divided by the bisector of the largest angle.

41. Find the area of the quadrilateral $ABCD$ if $AB = 78$ m., $BC = 104$ m., $CD = 50$ m., $AD = 120$ m., and $AC = 130$ m.

42. One diagonal of a rhombus is $\frac{8}{15}$ of the other and the difference of the diagonals is 14. Find the area and the perimeter of the rhombus.

43. A trapezoid is composed of a rhombus and an equilateral triangle. Each side of each figure is 16 inches. Find the area of the trapezoid.

44. Find the side of an equilateral triangle equal in area to the square whose diagonal is $15\sqrt{2}$.

45. Which of the figures in Ex. 44 has the smaller perimeter?

46. In a triangle whose base is 20 and whose altitude is 12, a line is drawn parallel to the base, bisecting the area of the triangle. Find the distance from the base to this parallel.

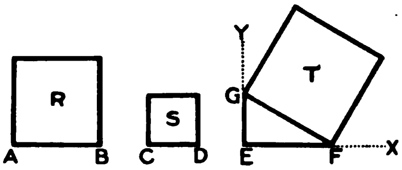
47. Two lines are drawn parallel to the base of a triangle whose base is 30 and altitude 18. These lines divide the area of the triangle into three equal parts. Find their distances from the vertex.

48. Around a rectangular lawn 30 yards $\times$ 20 yards is a drive 16 feet wide. How many square yards are there in the drive?

CONSTRUCTION PROBLEMS

PROPOSITION XVII. PROBLEM

388. To construct a square equal to the sum of two squares.

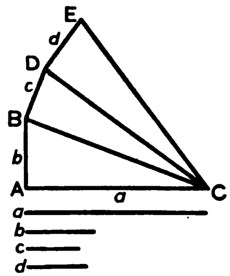


Given: (?). **Required:** (?). **Construction:** Construct a rt. $\angle E$, with sides EX and EY. On EX take $EF =$ to AB; on EY take $EG =$ to CD. Draw FG. On FG construct square T.

Statement: $T = R + S$. Q.E.F.

Proof: $T = R + S$ (377). Q.E.D.

PROPOSITION XVIII. PROBLEM



389. To construct a square equal to the sum of several squares.

Given: Squares whose sides are a, b, c, d.

Required: To construct a square = to $a^2 + b^2 + c^2 + d^2$.

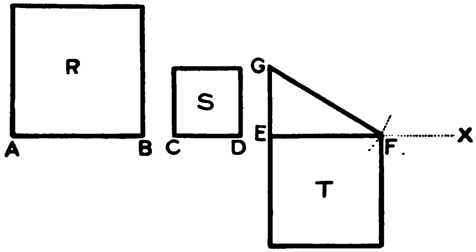
Construction: Construct a rt. $\angle$ whose sides are equal to a and b. Draw hypotenuse BC. At B erect a $\perp =$ to c and draw hypotenuse DC. At D erect a $\perp =$ to d, etc.

Statement: The square constructed on EC = the sum of the several given squares. Q.E.F.

$$\overline{EC}^2 = \overline{DC}^2 + d^2 \quad (334).$$
$$= \overline{BC}^2 + c^2 + d^2 \quad (?).$$
$$= a^2 + b^2 + c^2 + d^2 \quad (?). \quad \text{Q.E.D.}$$

Proposition XIX. Problem

390. To construct a square equal to the difference of two given squares.



Given: (?). **Required:** (?).

Construction: At one end of indefinite line, EX, erect $EG \perp$ to EX and $=$ to CD (a side of the **smaller** square, S). Using G as center and AB as radius, describe arc intersecting EX at F.

Draw GF. On EF construct square T.

Statement: $T = R - S$. Q.E.F. **Proof:** (?).

391. To construct a polygon similar to two given similar polygons and equal to their sum.

Construction: Like 388. **Proof:** (379).

392. To construct a polygon similar to two given similar polygons and equal to their difference.

Construction: Like 390. **Proof:** (380).

Ex. 1. Construct a right triangle whose area shall equal the area of a given square.

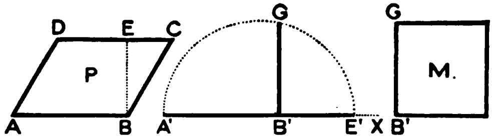
Ex. 2. Construct an isosceles triangle whose area shall equal the area of a given square.

Ex. 3. Construct an isosceles triangle equal in area to a given right triangle.

Ex. 4. Construct an isosceles triangle equal in area to any given triangle.

PROPOSITION XX. PROBLEM

393. To construct a square equal in area to a given parallelogram.



Given: (?). **Required:** (?) **Construction:** Construct a mean proportional between the base, AB, and the altitude, BE; on this mean proportional, $B'G$, construct a square, M.

Statement: Square $M =$ parallelogram P. Q.E.F.

Proof: $AB : B'G = B'G : BE$ (Const.).

$\therefore \overline{B'G}^2 = AB \cdot BE$ (280).

But $\overline{B'G}^2 =$ square M (358).

And $AB \cdot BE = \square P$ (359).

$\therefore$ square $M = \square P$ (Ax. 1).

Q.E.D.

394. To construct a square equal in area to a given triangle: Construct a mean proportional between half the base and the altitude, and proceed as in 393.

Ex. 1. Construct a square equal in area to a given right triangle.

Ex. 2. Construct a square equal in area to the sum of any two triangles; equal to their difference.

Ex. 3. Construct a square equal in area to the sum of two parallelograms; equal to their difference.

Ex. 4. Construct a right triangle equal in area to a given parallelogram, and with the same base.

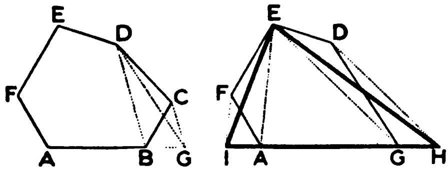
Ex. 5. Construct an isosceles triangle equal in area to a given parallelogram, and with the same base.

Ex. 6. Construct on the same base as a square, an isosceles triangle equal in area to the square.

Ex. 7. Construct a right triangle equal in area to the sum of two given squares.

Proposition XXI. Problem

395. To construct a triangle equal to a given polygon.



Given: Polygon AD. **Required:** To construct a $\triangle =$ to AD.

Construction: Draw a diagonal, BD, connecting any vertex (B) to the next but one (D). From the vertex between these (C), draw $CG \parallel$ to BD, meeting AB prolonged, at G. Draw DG. Repeat (2d figure) by drawing EG and $DH \parallel$ to EG, then EH. Repeat again by drawing AE, $FI \parallel$ to AE, then EI.

Statement: $\triangle\ IHE =$ polygon $ABCDEF$. Q.E.F.

Proof: In first figure, $\triangle\ BGD = \triangle\ BCD$ (367).

Add $\underline{\text{polygon } ABDEF = \text{polygon } ABDEF}$

$\therefore$ pentagon $AGDEF =$ polygon $ABCDEF$ (Ax. 2).

Also, in second figure, $\triangle\ GHE = \triangle\ GDE$ (367).

Add $\underline{\text{polygon } AGEF = \text{polygon } AGEF}$

$\therefore$ quadrilateral $AHEF =$ pentagon $AGDEF$ (Ax. 2).

Again, $\triangle\ AIE = \triangle\ AFE$ (367).

Add $\underline{\triangle\ AHE = \triangle\ AHE}$

$\therefore$ $\triangle\ IHE =$ quad. $AHEF$ (Ax. 2).

Hence $\triangle\ IHE =$ polygon $ABCDEF$ (Ax. 1).

Q.E.D.

396. To construct a square equal to a given polygon.

First, construct a $\triangle =$ to the polygon (by 395).

Second, construct a square $=$ to the $\triangle$ (by 394).

Ex. 1. Tell how to construct, by two methods, a square equal to a parallelogram.

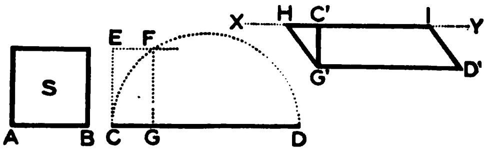
Ex. 2. How can rectilinear figures be added into a single figure?

PROPOSITION XXII. PROBLEM

397. To construct a parallelogram (or a rectangle) equal to a given square, and having:

I. The sum of its base and altitude equal to a given line.

II. The difference of its base and altitude equal to a given line.



I. Given: Square S and line CD.

Required: To construct a $\square =$ to S; base + altitude $= CD$.

Construction: On CD as a diameter describe a semicircle. At C erect $CE \perp$ to CD and $=$ to AB. Through E draw $EF \parallel$ to CD, meeting the circumference at F. Draw $FG \perp$ to CD. Take $G'D' =$ to GD and draw $XY \parallel$ to $G'D'$ at the distance from it $=$ to CG. On XY take $HI = GD$. Draw HG' and ID'.

Statement: $\square\ G'D'IH = S$, and base + alt. $= CD$. Q.E.F.

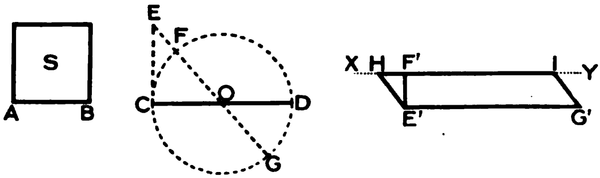
Proof:	$G'D'IH$ is a $\square$	(129).
Also	$GD \cdot GC = \overline{FG}^2$	(332).
But	$GD \cdot GC = \square\ G'D'IH$	(359).
And	$\overline{FG}^2 = \overline{EC}^2 = \overline{AB}^2 =$ square S	(124, 358).
Substituting,	$\square\ G'D'IH =$ square S	(Ax. 6).
Also	$G'D' + G'C' = CD$	(Ax. 4). Q.E.D.

Historical Note. The following extract from Nicolay and Hay's *Life of Abraham Lincoln* should be of interest to the student:

"His wider knowledge of men and things had shown him a certain lack in himself of the power of close and sustained reasoning. To remedy this defect, he applied himself, after his return from Congress, to such works upon logic and mathematics as he fancied would be serviceable. Devoting himself with dogged energy to the task in hand, he soon learned by heart six books of the propositions of Euclid, and he retained through life an intimate knowledge of the principles they contain."

II. Given: Square S and line CD.

Required: To construct a ▱ = to S; base − altitude = to CD.



Construction: On CD as diameter, describe a ⊙, O. At C erect $CE \perp$ to CD and = to AB. Draw $EFOG$ meeting ⊙ at F and G. Take $E'G'$ = to EG and draw $XY \parallel$ to $E'G'$ at a distance from it = to EF. On XY take HI = to EG. Draw HE' and IG'.

Statement: ▱ $E'G'IH = S$, and base − alt. = CD. Q.E.F.

Proof: $E'G'IH$ is a ▱ (129).

EC is tangent to ⊙ O (202).

$\therefore EG \cdot EF = \overline{EC}^2$ (324).

But $EG \cdot EF =$ ▱ $E'G'IH$ (359).

And $\overline{EC}^2 = \overline{AB}^2 =$ square S (358).

$\therefore$ ▱ $E'G'IH =$ square S (Ax. 6).

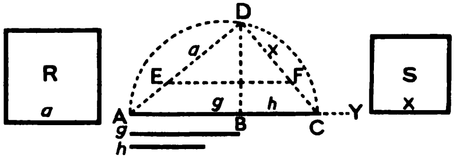
Also $E'G' - E'F' = FG = CD$ (190). Q.E.D.

398. To find two lines whose product is given:

I. If their sum is also given.
II. If their difference is also given. } [The same as 397.]

PROPOSITION XXIII. PROBLEM

399. To construct a square having a given ratio to a given square.



Given: Square R, and lines g and h.

Required: To construct a square such that it (the unknown square) : square $R = h : g$.

Construction: On an indefinite line AY take $AB =$ to g, and $BC =$ to h. On AC as diameter describe a semicircle. At B erect $BD \perp$ to AC, meeting arc at D. Draw AD and CD. On AD take $DE =$ to a, and draw $EF \parallel$ to AC, meeting DC at F. Using $DF = x$, as a side, construct square S.

Statement: $S : R = h : g$. Q.E.F.

Proof: $\angle ADC$ is a rt. $\angle$ (240).

$$\therefore \frac{\overline{CD}^2}{\overline{AD}^2} = \frac{h}{g} \qquad (381).$$

But $$\frac{x}{a} = \frac{CD}{AD} \qquad (294).$$

$$\therefore \frac{x^2}{a^2} = \frac{\overline{CD}^2}{\overline{AD}^2} \qquad (287).$$

$$\therefore \frac{x^2}{a^2} = \frac{h}{g} \qquad (\text{Ax. } 1).$$

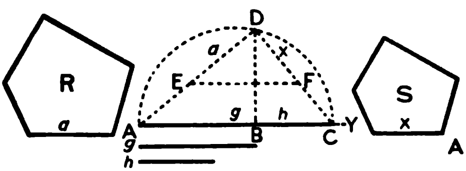
But $x^2 =$ square S, and $a^2 =$ square R (358).

Substituting, square S : square $R = h : g$ (Ax. 6).

Q.E.D.

PROPOSITION XXIV. PROBLEM.

400. To construct a polygon similar to a given polygon and having a given ratio to it.

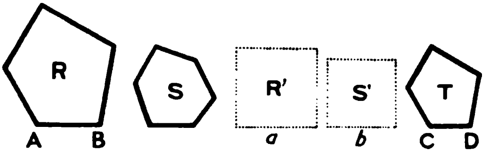


Given: (?). **Required:** (?). **Construction** and **Statement** are the same as for Proposition XXIII.

Proof: Very much like the proof of Proposition XXIII.

PROPOSITION XXV. PROBLEM

401. To construct a polygon similar to one given polygon and equal in area to another.



Given: Polygons R and S. **Required:** (?).

Construction: Construct squares $R' = R$, and $S' = S$ (by 396). Find a fourth proportional to a, b, and AB. This is CD. Upon CD, homologous to AB, construct T similar to R.

Statement: $T = S$. Q.E.F.

Proof: $\frac{R}{T} = \frac{\overline{AB}^2}{\overline{CD}^2}$ (376).

$\frac{a}{b} = \frac{AB}{CD}$ (Const.).

$\therefore \frac{a^2}{b^2} = \frac{\overline{AB}^2}{\overline{CD}^2}$ (287).

$\therefore \frac{R}{T} = \frac{a^2}{b^2}$ (Ax. 1).

Now $a^2 = R' = R$; $a^2 = S' = S$ (358 & Const.).

Substituting, $\frac{R}{T} = \frac{R}{S}$ (Ax. 6).

$\therefore T = S$ (Ax. 3).

Q.E.D.

NOTE. By means of this proposition we are able to maintain the size of a rectilinear figure, but change its shape to any desired form. Thus we can construct an equilateral triangle equal to a given square. The pupil will explain.

ORIGINAL CONSTRUCTIONS

It is required:

1. To construct a right triangle equal to a given parallelogram.

2. To construct a square equal to the sum of two given parallelograms.

3. To construct a square equal to the difference of two given parallelograms.

4. To construct a square equal to the sum of several given right triangles.

5. To construct a square equal to the sum of several given parallelograms.

6. To construct a square equal to the sum of several given triangles.

7. To construct a square equal to the sum of several given polygons.

8. To construct a square equal to the difference of two given polygons.

9. To construct a square equal to three times a given square; seven times a given square.

10. To construct a right triangle equal to the sum of several given triangles.

11. To construct a right triangle equal to the difference of any two given triangles; of any two given parallelograms.

12. To construct a square equal to a given trapezoid; equal to a given trapezium.

13. To construct a square equal to a given hexagon.

14. To construct a rectangle equal to a given triangle, having given its perimeter.

15. To construct an isosceles right triangle equal to a given triangle.

16. To construct a square equal to a given rhombus.

17. To construct a rectangle equal to a given trapezium, and having its perimeter given.

18. To find a line whose length shall be $\sqrt{2}$ units. [See 388.]

19. To find a line whose length shall be $\sqrt{3}$ units.

20. To find a line whose length shall be $\sqrt{11}$ units.

21. To find a line whose length shall be $\sqrt{7}$ units.

22. To find a line whose length shall be $\sqrt{30}$ units.

23. To construct a square which shall be $\frac{3}{5}$ of a given square.

24. To construct a square which shall be $\frac{4}{5}$ of a given square.

25. To construct a polygon which shall be $\frac{2}{7}$ of a given polygon, and similar to it.

26. To construct a square which shall have to a given square the ratio $\sqrt{3}:4$; the ratio $4:\sqrt{3}$.

27. To draw through a given point, within a parallelogram, a line which shall bisect the parallelogram.

28. To construct a rectangle equal to a given trapezoid, having given the difference of its base and altitude.

29. To construct a triangle similar to two given similar triangles and equal to their sum.

30. To construct a triangle similar to a given triangle and equal to a given square. [See 401.]

31. To construct a triangle similar to a given triangle and equal to a given parallelogram.

32. To construct a square having twice the area of a given square. [Two methods.]

33. To construct a square having $3\frac{1}{2}$ times the area of a given square.

34. To construct an isosceles triangle equal to a given triangle and upon the same base.

35. To construct a triangle equal to a given triangle, having the same base, and also having a given angle adjoining this base.

36. To construct a parallelogram equal to a given parallelogram having the same base and also having a given angle adjoining the base.

37. To draw a line that shall be perpendicular to the bases of a parallelogram and that shall bisect the parallelogram.

38. To construct an equilateral triangle equal to a given triangle. [See 401.]

39. To trisect (divide into three equal parts) the area of a triangle, by lines drawn from one vertex.

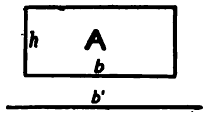
40. To construct a square equal to $\frac{2}{3}$ of a given pentagon.

41. To construct an isosceles trapezoid equal to a given trapezoid.

42. To construct an equilateral triangle equal to the sum of two given equilateral triangles.

43. To construct an equilateral triangle equal to the difference of two given equilateral triangles.

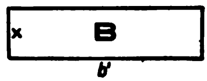
44. To construct upon a given base a rectangle that shall be equal to a given rectangle.



Analysis: Let us call the unknown altitude x. Then $b \cdot h = b' \cdot x$ (?). Hence, $b' : b = h : x$ (?).

That is, the *unknown altitude* is a fourth proportional to *the given base, the base of the given rectangle*, and *the altitude of the given rectangle.*

Construction: Find a fourth proportional, x, to b', b and h. Construct a rectangle having base $= b'$ and alt. $= x$.



Statement: This rectangle, $B = A$.

Proof: $b' : b = h : x$ (Const.). $\therefore$ $b'x = bh$ (?). But $b'x =$ the area of B (?), etc.

45. To construct a rectangle that shall have a given altitude and be equal to a given rectangle.

46. To construct a triangle upon a given base that shall be equal to a given triangle.

47. To construct a triangle that shall have a given altitude and be equal to a given triangle.

48. To construct a rectangle that shall have a given base, and shall be equal to a given triangle.

49. To construct a triangle that shall have a given base, and be equal to a given rectangle.

50. To construct a triangle that shall have a given base, and be equal to a given polygon.

51. To construct the problems 45, 46, 47, 48, 49, substituting "parallelogram" in each case for the figure to be constructed.

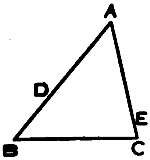
52. To construct upon a given hypotenuse, a right triangle equal to a given triangle.

53. To construct upon a given hypotenuse, a right triangle equal to a given square.

54. To construct (*a*) a triangle which shall have a given base, a given adjoining angle, and be equal to a given triangle; (*b*) a triangle equal to a given square; (*c*) a triangle equal to a given polygon.

55. To construct (*a*) a parallelogram which shall have a given base, a given adjoining angle, and be equal to a given parallelogram; (*b*) a parallelogram equal to a given triangle; (*c*) a parallelogram equal to a given polygon.

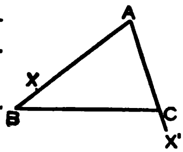
56. To construct a line, DE, from D, a given point in AB of triangle ABC, so that DE bisects the triangle.



Analysis: After DE is drawn, $\triangle ABC = 2 \triangle ADE$ (Hyp.). But $\triangle ABC : \triangle ADE = AB \cdot AC : AD \cdot AE$ (?). Hence, $AB \cdot AC = 2\,(AD \cdot AE)$ (Ax. 6).

$\therefore 2\,AD : AB = AC : x$ (?). Thus x (that is, AE) is a fourth proportional to three given lines.

57. To draw a line meeting two sides of a triangle and forming an isosceles triangle equal to the given triangle.



Analysis: Suppose AX is a leg of the required isosceles $\triangle$. $\therefore \triangle ABC : \triangle AXX' = AB \cdot AC : AX \cdot AX'$. But the $\triangle$s are equal and $AX = AX'$ (Hyp.).

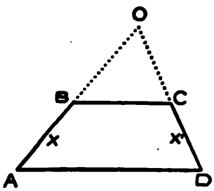
Hence, $AB \cdot AC = \overline{AX}^2$. $\therefore AX$ is a mean proportional between AB and AC.

58. To draw a line parallel to the base of a triangle which shall bisect the triangle.

59. To draw a line meeting two sides of a triangle forming an isosceles triangle equal to half the given triangle.

60. To draw a line parallel to the base of a triangle forming a triangle equal to one third of the original triangle.

61. To draw a line parallel to the base of a trapezoid so that the area is bisected.



Analysis: $\triangle OXX' = \frac{1}{2}(\triangle OAD + \triangle OBC)$ and is similar to $\triangle OBC$.

62. To construct two lines parallel to the base of a triangle, that shall trisect the area of the triangle.

63. To construct a triangle, having given its angles and its area.

Analysis: The required $\triangle$ is similar to any $\triangle$ containing the given $\angle$s. The given area may be a square. This reduces the problem to 401.

64. To find two straight lines in the ratio of two given polygons.

BOOK V

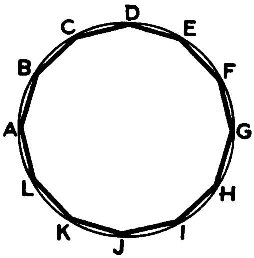
REGULAR POLYGONS. CIRCLES

THEOREMS AND DEMONSTRATIONS

402. A **regular polygon** is a polygon which is equilateral and equiangular.

PROPOSITION I. THEOREM

403. An equilateral polygon inscribed in a circle is regular.



Given: AG, an equilateral inscribed polygon.

To Prove: AG is regular.

Proof: chord AB = chord BC = chord CD = etc. (Hyp.).

$\therefore$ arc AB = arc BC = arc CD = etc. (196).

$\therefore$ arc AC = arc BD = arc CE = etc. (Ax. 3).

$\therefore \angle ABC = \angle BCD = \angle CDE$ = etc. (239).

That is, the polygon is equiangular.

$\therefore$ the polygon is regular (402). Q. E. D.

404. COROLLARY. **If the circumference of a circle is divided into any number of equal arcs, and the chords of these arcs are drawn, they form an inscribed polygon.**

Given: Arc AB = arc BC = arc CD = etc. and chords AB, etc.

To Prove: Polygon AG is a regular polygon.

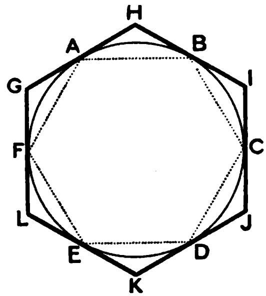
Proof: Chords AB, BC, CD, etc. are all equal. (197).

$\therefore$ the polygon is regular (403). Q. E. D.

405. COROLLARY. **If chords are drawn joining the alternate vertices of an inscribed regular polygon (having an even number of sides), another inscribed regular polygon is formed.**

Proposition II. Theorem

406. If the circumference of a circle is divided into any number of equal parts, and tangents are drawn, at the several points of division, they form a circumscribed regular polygon.



Given: Arcs AB, BC, CD, etc. all equal; and GH, HI, IJ, etc. tangents at A, B, C, etc.

To Prove: Polygon HK is regular.

Proof: Draw chords AB, BC, CD, etc.

In ▲ ABH, BCI, CDJ, etc. $AB = BC = CD$, etc. (197).

$\angle HAB = \angle HBA = \angle IBC = \angle ICB = \angle JCD$, etc. (237).

∴ these ▲ are congruent and isosceles (76; 114).

∴ $\angle H = \angle I = \angle J =$ etc. (27).

That is, polygon HK is equiangular.

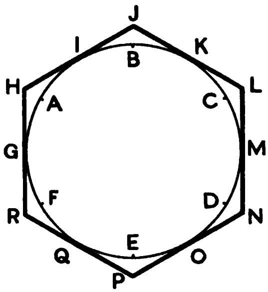
Also $AH = HB = BI = IC = CJ$, etc. (24 or 206; 27).

∴ $HI = IJ = JK =$ etc. (Ax. 3).

That is, polygon HK is equilateral.

∴ polygon HK is regular (402).

Q.E.D.



407. Corollary. If the circumference of a circle is divided into any number of equal parts and tangents are drawn at their midpoints, they form a circumscribed regular polygon.

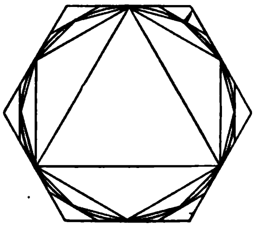
Given: (?) **To Prove:** (?).

Proof: Arcs AB, BC, CD, etc. are all equal (Hyp.).

Also arcs AI, IB, BK, KC, CM, etc. are all equal (Ax. 3).

∴ arcs IK, KM, MO, etc. are all equal (Ax. 3).

Therefore the polygon is regular (406). Q.E.D.



408. Corollary. If the vertices of an inscribed regular polygon are joined to the midpoints of the arcs subtended by the sides, another inscribed regular polygon is formed (having double the number of sides).

409. Corollary. If tangents are drawn at the midpoints of the arcs between adjacent points of contact of the sides of a circumscribed regular polygon, another circumscribed regular polygon is formed having double the number of sides. (?).

410. Corollary. The perimeter of an inscribed regular polygon is less than the perimeter of an inscribed regular polygon having twice as many sides, and the perimeter of a circumscribed regular polygon is greater than the perimeter of a circumscribed regular polygon having twice as many sides.

(Ax. 12.)

Ex. 1. If in a regular dodecagon (Fig. of 403) all the diagonals are drawn from vertex A, how many degrees are there between adjacent pairs of diagonals, at A?

Ex. 2. In the figure of 406, how many degrees are there in each of the three angles at A? Prove in three ways that there are 120° in $\angle G$. If radii were drawn to the vertices of the inscribed polygon, what kind of triangles would be formed?

Ex. 3. Write a formula for finding the number of degrees in each angle of a regular polygon.

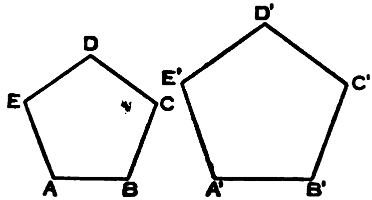
Ex. 4. Are the following figures regular polygons: a square? an isosceles triangle? a rectangle? a rhombus? an equilateral triangle?

Ex. 5. In the figure of 406, prove that a line from the center of the circle OI is the perpendicular bisector of chord BC. Then prove triangles OBI and OBM similar (M being midpoint of BC). Hence prove $OI : OB = OB : OM$.

Ex. 6. Can you explain how to divide a circle into three equal parts? into four equal parts?

Proposition III. Theorem

411. Two regular polygons having the same number of sides are similar.



Given: Regular n-gons AD and $A'D'$.

To Prove: They are similar.

Proof:
$$\angle A = \frac{(n-2)\,180^\circ}{n} \qquad (155).$$
$$\angle A' = \frac{(n-2)\,180^\circ}{n} \qquad (?).$$
$$\therefore \angle A = \angle A' \qquad (\text{Ax. } 1).$$
Similarly, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

That is, these polygons are mutually equiangular.

Now $AB = BC = CD =$ etc.; $A'B' = B'C' = C'D' =$ etc. (402).

$\therefore AB : A'B' = BC : B'C' = CD : C'D' =$ etc. (Ax. 3).

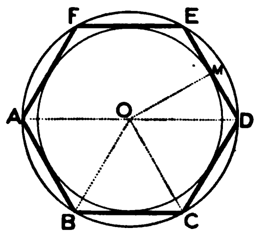
That is, the homologous sides are proportional.

Therefore the polygons are similar (301).

Q.E.D.

Proposition IV. Theorem

412. A circle can be circumscribed about, and a circle can be inscribed in, any regular polygon.



Given: Regular polygon $ABCDEF$.

To Prove: I. A circle can be circumscribed about the polygon.

II. A circle can be inscribed in the polygon.

Proof: I. Through three consecutive vertices, A, B, and C, describe a circumference, whose center is O. Draw radii OA, OB, OC, and draw line OD.

In $\triangle$ AOB and COD, $AB = CD$ (402), $BO = CO$ (187).

Also $\angle ABC = \angle BCD$ (402), $\angle OBC = \angle OCB$ (55).

Subtracting, $\angle ABO = \angle OCD$ (Ax. 2).

$\therefore$ $\triangle AOB$ is congruent to $\triangle COD$ (52).

$\therefore AO = OD$ (27).

Hence the arc passes through D, and in like manner it may be proved that it passes through E and F.

That is, a circle can be circumscribed about the polygon.

II. AB, BC, CD, DE, etc. are equal chords (402).

$\therefore$ they are equally distant from the center (208).

That is, a circle described, using O as a center and OM as a radius, will touch every side of the polygon. (202).

Hence a circle can be inscribed (221). Q.E.D.

413. The **radius** of a regular polygon is the radius of the circumscribed circle. The radius of the inscribed circle is called the **apothem**. The **center** of a regular polygon is the common center of the circumscribed and inscribed circles.

414. The **central angle** of a regular polygon is the angle included between two radii drawn to the ends of a side.

415. THEOREM. **Each central angle of a regular n-gon $= \dfrac{360°}{n}$.**

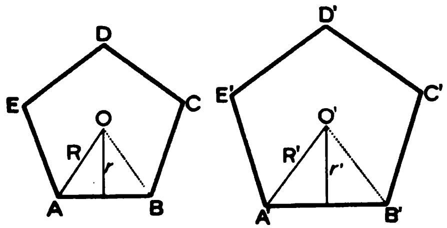
416. COROLLARY. **Each exterior angle of a regular n-gon $= \dfrac{360°}{n}$.**

417. COROLLARY. **The radius drawn to any vertex of a regular polygon bisects the angle at the vertex** (95).

418. COROLLARY. **The central angles of regular polygons having the same number of sides are equal.**

PROPOSITION V. THEOREM

419. The perimeters of two regular polygons having the same number of sides are to each other as their radii and also as their apothems.



Given: Regular n-gons, EC, with perimeter P, radius R, apothem r; and $E'C'$ with perimeter P', radius R', apothem r'.

To Prove: $P : P' = R : R' = r : r'$.

Proof: Draw radii OB and $O'B'$.

In $\triangle$s AOB and $A'O'B'$, $\angle AOB = \angle A'O'B'$ (418).

$AO = BO$ (187).

$A'O' = B'O'$ (?).

$\therefore \dfrac{AO}{A'O'} = \dfrac{BO}{B'O'}$ (Ax. 3).

$\therefore \triangle AOB$ is similar to $\triangle A'O'B'$ (306).

$\therefore \dfrac{AB}{A'B'} = \dfrac{R}{R'} = \dfrac{r}{r'}$ (313).

But these polygons are similar (411).

$\therefore \dfrac{P}{P'} = \dfrac{AB}{A'B'}$ (317).

$\therefore \dfrac{P}{P'} = \dfrac{R}{R'} = \dfrac{r}{r'}$ (Ax. 1).

Q.E.D.

PROPOSITION VI. THEOREM

420. The areas of two regular polygons having the same number of sides are to each other as the squares of their radii and also as the squares of their apothems.

Given: Regular n-gons, EC whose area is K, radius R, apothem r; and $E'C'$ whose area is K', radius R', apothem r'.

To Prove: $K : K' = R^2 : R'^2 = r^2 : r'^2$.

Proof: As in 419, $\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}$.

Then $\frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$ (287).

But these polygons are similar (411).

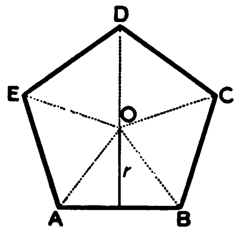
$\therefore \frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$ (376).

$\therefore \frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$ (Ax. 1).

Q.E.D.

PROPOSITION VII. THEOREM

421. The area of a regular polygon is equal to half the product of the perimeter by the apothem.



Given: (?).

To Prove: (?).

Proof: Draw radii to all the vertices, forming several isosceles triangles.

$$\left.\begin{aligned} \text{Area of } \triangle AOB &= \tfrac{1}{2} AB \cdot r \\ \text{Area of } \triangle BOC &= \tfrac{1}{2} BC \cdot r \\ \text{Area of } \triangle COD &= \tfrac{1}{2} CD \cdot r \\ \text{etc., etc.} \end{aligned}\right\} \quad (364).$$

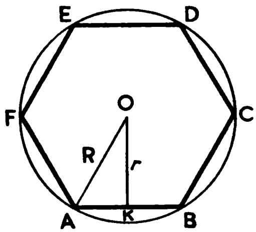
Area of polygon $= \frac{1}{2}(AB + BC + CD + \text{etc.}) \cdot r$ (Ax. 2).

Substituting, area $= \frac{1}{2} P \cdot r$ (Ax. 6).

Q.E.D.

Proposition VIII. Theorem

422. If the number of sides of an inscribed regular polygon is increased indefinitely, the apothem approaches the radius as a limit.



Given: Polygon FC inscribed in $\odot O$; apothem $= r$; radius $= R$.

To Prove: That as the number of sides is indefinitely increased, r approaches R as a limit.

Proof: In the $\triangle AOK$,

$$R < r + AK \qquad \text{(Ax. 12).}$$

Or $$R - r < AK \qquad \text{(Ax. 7).}$$

Now, as the number of sides of the polygon is indefinitely increased, AB is indefinitely decreased.

Hence $\frac{1}{2} AB$, or AK, approaches zero as a limit.

$\therefore$ $R - r$ approaches zero (because $R - r < AK$).

That is, r approaches R as a limit (227). Q.E.D.

NOTE. It is evident that if the difference between two variables approaches zero, either

(1) one is approaching the other as a limit; or

(2) both are approaching some third quantity as their limit.

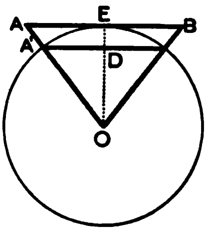
423. Theorem. The circumference of a circle is less than the perimeter of any circumscribed polygon.

By drawing tangents at the midpoints of the included arcs of a circumscribed polygon, another circumscribed polygon is formed; the perimeter of this polygon is less than the perimeter of the given polygon. This can be continued indefinitely, decreasing the perimeter of the polygons. Hence there can be no circumscribed polygon whose perimeter can be the least of all such polygons; because, by increasing the number of sides, the perimeter is lessened. Hence the circumference must be less than the perimeter of any circumscribed polygon.

424. Theorem. If the number of sides of an inscribed regular polygon and of a circumscribed regular polygon is indefinitely increased.

I. The perimeter of each polygon approaches the circumference of the circle as a limit.

II. The area of each polygon approaches the area of the circle as a limit.



Given: A circle O, whose circumference is C and whose area is S; AB and $A'B'$, sides of regular circumscribed and inscribed polygons, having the same number of sides; P and P', their perimeters; K and K', their areas.

To Prove: That if the number of sides is indefinitely increased:

I. P approaches C and P' approaches C as limit.

II. K approaches S and K' approaches S as limit.

Proof: I. The polygons are similar (411).

$$\therefore \frac{P}{P'} = \frac{OE}{OD} \quad (419).$$

Now, if the number of sides of these polygons is indefinitely increased, OD approaches OE (422).

Hence $\frac{OE}{OD}$ approaches 1. That is, $\frac{P}{P'}$ approaches 1, or P and P' approach equality; that is, they approach the same constant as a limit.

But $P > C$ and $C > P'$ and C is constant.

Hence P approaches C and P' approaches C. Q.E.D.

II. $$\frac{K}{K'} = \frac{\overline{OE}^2}{\overline{OD}^2} \quad (420).$$

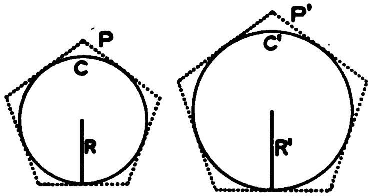
If the number of sides of these polygons is indefinitely increased, $\overline{OD}^2$ approaches $\overline{OE}^2$, and thus $\frac{\overline{OE}^2}{\overline{OD}^2}$ approaches unity.

(The argument continues the same as in I.)

NOTE. The theorems of 423 and 424 are considered so evident, and rigorous proofs (as in the case of the demonstrations for many fundamental principles in mathematics) are so difficult for young students to comprehend, that it is advisable to omit the profound demonstrations and insert only simple explanations.

PROPOSITION IX. THEOREM

425. The circumferences of two circles are to each other as their radii.



Given: Two Ⓢ whose radii are R and R' and circumferences, C and C' respectively.

To Prove: $C : C' = R : R'$.

Proof: Circumscribe regular polygons (having the same number of sides) about these Ⓢ and let P and P' denote their perimeters. Then $P : P' = R : R'$ (419).

Hence $P \cdot R' = P' \cdot R$ (?).

Now suppose the number of sides of these polygons to be indefinitely increased,

P approaches C (424).

P' approaches C' (?).

$\therefore$ $P \cdot R'$ approaches $C \cdot R'$.

Also $P' \cdot R$ approaches $C' \cdot R$.

Hence $C \cdot R' = C' \cdot R$ (229).

Therefore $C : C' = R : R'$ (281).

Q.E.D.

426. THEOREM. **The ratio of any circumference to its diameter is constant for all circles. That is, any circumference divided by its diameter is the same as any other circumference divided by its diameter.**

Proof: $$\frac{C}{C'} = \frac{R}{R'} \quad (425).$$

But $$\frac{R}{R'} = \frac{\frac{1}{2}D}{\frac{1}{2}D'} = \frac{D}{D'} \quad (\text{Ax. } 6).$$

$$\therefore \frac{C}{C'} = \frac{D}{D'} \qquad \text{(Ax. 1).}$$

$$\therefore \frac{C}{D} = \frac{C'}{D'} \qquad \text{(282).}$$

That is, $\dfrac{\text{circumference}}{\text{diameter}} = $ a constant for all ⓢ. Q.E.D.

427. Definition of π (pi). The constant ratio of a circumference to its diameter is called π. That is, $\dfrac{C}{D} = \pi$.

The numerical value of π is 3.141592, or $3\frac{1}{7}$, approximately. (This is computed in 453.)

428. Formula. Let $C =$ the circumference of a circle with radius R. Then $\dfrac{C}{2R} = \pi$. (427).

$$\therefore \boldsymbol{C = 2\pi R} \qquad \text{(Ax. 3).}$$

Ex. 1. Find the circumference of a circle the radius of which is 12 in.

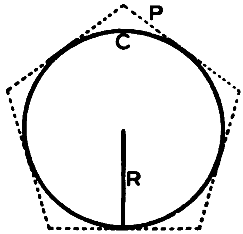
Ex. 2. Find the radius of a circle the circumference of which is 66 feet.

Historical Note. Gottfried Wilhelm von Leibnitz, a German philosopher, mathematician, and man of affairs, was born in 1646 and died in 1716. He could read Latin easily at 12, and wrote some Latin verse. While at the University of Leipsic he became acquainted with Francis Bacon, Kepler, Galileo, and Descartes, modern thinkers who had revolutionized science and philosophy. He resolved to study mathematics, but not until he had reached his majority did he throw himself into deep mathematical research. It was while he was living in Paris and Mainz that he announced his imposing discoveries in natural philosophy, mathematics, mechanics, optics, hydrostatics, pneumatics, and nautical science. In mathematics, he was the discoverer of the differential and integral calculus. He possessed a marvelous ability for rapid and continuous work. Even in traveling his time was employed in solving mathematical problems. He is described as moderate in habits, quick of temper, charitable in judgment of others, tolerant of differences of opinion, but impatient of contradiction on small matters and desirous of honor.

LEIBNITZ

PROPOSITION X. THEOREM

429. The area of a circle is equal to half the product of its circumference by its radius.



Given: ⊙ with circumference C, area S, radius R.

To Prove: $S = \frac{1}{2} C \cdot R$.

Proof: Circumscribe a regular polygon about the circle; denote its area by K and perimeter by P.

Now $K = \frac{1}{2} P \cdot R$ (421).

Suppose the number of sides of the polygon is indefinitely increased. K approaches S, and P approaches C (424).

$\frac{1}{2} P \cdot R$ approaches $\frac{1}{2} C \cdot R$ as a limit.

Hence $S = \frac{1}{2} C \cdot R$ (229). Q.E.D.

430. FORMULA. Let $S =$ the area of a circle whose circumference $= C$, and whose radius $= R$.

Then $S = \frac{1}{2} C \cdot R$ (429).

But $C = 2\pi R$ (428).

Substituting, $\boldsymbol{S = \pi R^2}$ (Ax. 6).

Ex. 1. Find the area of a circle the radius of which is 8 in.

Ex. 2. Find the radius of a circle the area of which is 500 sq. ft.

431. COROLLARY. **The areas of two circles are to each other as the squares of their radii, and as the squares of their diameters.**

To Prove: $S : S' = R^2 : R'^2 = D^2 : D'^2$.

Proof: $S = \pi R^2$, and $S' = \pi R'^2$ (430).

Dividing, $\frac{S}{S'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2}$ (Ax. 3).

Now $\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2} D)^2}{(\frac{1}{2} D')^2} = \frac{\frac{1}{4} D^2}{\frac{1}{4} D'^2} = \frac{D^2}{D'^2}$ (Ax. 6). Q.E.D.

432. Corollary. **The area of a sector is the same part of the circle as its central angle is of 360°.** (Ax. 1.)

433. Formula. An arc : circum. = central $\angle$: 360° (231).

$$\therefore \textbf{arc} : 2\pi R = \angle : 360°.$$

Note. If any two of the three quantities, arc, R, $\angle$, are known, the remaining one can be found by this proportion.

434. Formula. Sector : area of $\odot$ = central $\angle$: 360° (432).

$$\therefore \textbf{sector} : \pi R^2 = \angle : 360°.$$

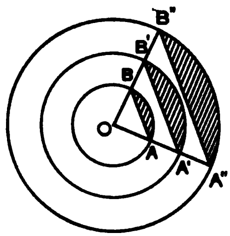
Note. If any two of the three quantities, sector, R, $\angle$, are known, the remaining one can be found by this proportion.

435. Formula. Sector : area of $\odot$ = arc : circum. (Ax. 1).

$$\therefore \text{sector} : \pi R^2 = \text{arc} : 2\pi R \quad \text{(Ax. 6)}.$$

$$\therefore \textbf{sector} = \tfrac{1}{2} R \cdot \textbf{arc} \quad \textbf{(280)}.$$

436. **Similar arcs, similar sectors,** and **similar segments** are those which correspond to equal central angles, in unequal circles.



Thus, AB, $A'B'$, $A''B''$ are similar arcs; AOB, $A'OB'$, and $A''OB''$ are similar sectors; and the shaded segments are similar segments.

437. Theorem. **Similar arcs are to each other as their radii.**

Given: Arcs whose lengths are a and a', radii R and R'.

To Prove: $a : a' = R : R'$.

Proof:
$$\frac{a}{2\pi R} = \frac{\angle}{360°}, \text{ and } \frac{a'}{2\pi R'} = \frac{\angle}{360°} \quad (433).$$

$$\therefore \frac{a}{2\pi R} = \frac{a'}{2\pi R'} \quad \text{(Ax. 1)}.$$

$$\therefore \frac{a}{a'} = \frac{2\pi R}{2\pi R'} = \frac{R}{R'} \quad (282).$$

Q.E.D.

438. Theorem. Similar sectors are to each other as the squares of their radii.

Given: Sectors whose areas are T and T', radii R and R'.

To Prove: $T : T' = R^2 : R'^2$.

Proof:
$$\frac{T}{\pi R^2} = \frac{\angle}{360}, \text{ and } \frac{T'}{\pi R'^2} = \frac{\angle}{360} \quad (434).$$

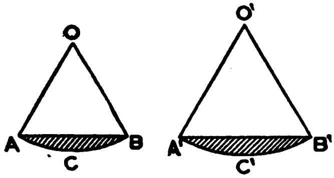
$$\therefore \frac{T}{\pi R^2} = \frac{T'}{\pi R'^2} \quad (\text{Ax. } 1).$$

$$\therefore \frac{T}{T'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2} \quad (282).$$

Q.E.D.

Proposition XI. Theorem

439. Similar segments are to each other as the squares of their radii.



Given: Similar segments ABC and $A'B'C'$.

To Prove: Segment ABC : segment $A'B'C' = R^2 : R'^2$.

Proof: $\triangle$s AOB and $A'O'B'$ are similar (306).

$$\therefore \frac{\triangle AOB}{\triangle A'O'B'} = \frac{R^2}{R'^2} \quad (375).$$

Also
$$\frac{\text{sector } OACB}{\text{sector } O'A'C'B'} = \frac{R^2}{R'^2} \quad (438).$$

$$\therefore \frac{\text{sector } OACB}{\text{sector } O'A'C'B'} = \frac{\triangle AOB}{\triangle A'O'B'} \quad (\text{Ax. } 1).$$

$$\therefore \frac{\text{sector } OACB}{\triangle AOB} = \frac{\text{sector } O'A'C'B'}{\triangle A'O'B'} \quad (282).$$

$$\therefore \frac{\text{sector } OACB - \triangle AOB}{\triangle AOB} = \frac{\text{sector } O'A'C'B' - \triangle A'O'B'}{\triangle A'O'B'} \quad (285).$$

That is,
$$\frac{\text{segment } ABC}{\triangle AOB} = \frac{\text{segment } A'B'C'}{\triangle A'O'B'} \quad (\text{Ax. } 6).$$

$$\therefore \frac{\text{segment } ABC}{\text{segment } A'B'C'} = \frac{\triangle AOB}{\triangle A'O'B'} = \frac{R^2}{R'^2} \quad (282).$$

Q.E.D.

ORIGINAL EXERCISES (THEOREMS)

1. The central angle of a regular polygon is the supplement of the angle of the polygon.

2. An equiangular polygon inscribed in a circle is regular (if the number of its sides is odd).

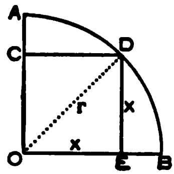
3. An equiangular polygon circumscribed about a circle is regular. [Draw radii and apothems.]

4. The sides of a circumscribed regular polygon are bisected at the points of contact.

5. The diagonals of a regular pentagon are equal.

6. The diagonals drawn from any vertex of a regular n-gon divide the angle at that vertex into $n-2$ equal parts.

7. If a regular polygon is inscribed in a circle, and another regular polygon having the same number of sides is circumscribed about it, the radius of the circle is a mean proportional between the apothem of the inner and the radius of the outer polygon.



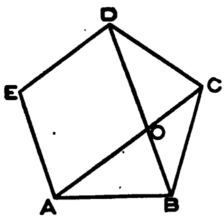
8. The area of the square inscribed in a sector the central angle of which is a right angle, is equal to half the square of the radius.

[Find x^2, the area of $OEDC$.]

9. The apothem of an equilateral triangle is one third the altitude of the triangle.

10. The chord that bisects a radius of a circle at right angles is the side of the inscribed equilateral triangle.

[Prove that the central $\angle$ subtended is 120°.]



11. If $ABCDE$ is a regular pentagon, and diagonals AC and BD are drawn, meeting at O:

(*a*) $AO = AB$.
(*b*) AO is $\parallel$ to ED.
(*c*) $\triangle BOC$ is similar to $\triangle BDC$.
(*d*) $\angle ACB = 36°$.
(*e*) AC is divided into mean and extreme ratio at O.

12. The altitude of an equilateral triangle is three fourths the diameter of the circumscribed circle.

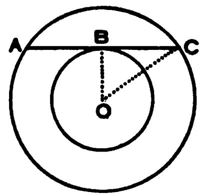
13. The apothem of an inscribed regular hexagon equals half the side of an inscribed equilateral triangle.

14. The area of a circle is four times the area of another circle described upon its radius as a diameter.

15. The area of an inscribed square is half the area of the circumscribed square.

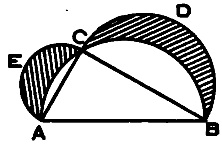
16. An equilateral polygon circumscribed about a circle is regular (if the number of its sides is odd).

17. The sum of the circles described upon the legs of a right triangle as diameters is equal to the circle described upon the hypotenuse as a diameter.



18. A circular ring (the area between two concentric circles) is equal to the circle described upon the chord of the larger circle, which is tangent to the less, as a diameter.

Proof: Draw radii OB, OC. $\triangle OBC$ is rt. $\triangle$ (?); and $\overline{OC}^2 - \overline{OB}^2 = \overline{BC}^2$ (?). Etc.



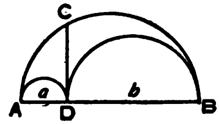
19. If semicircles are described upon the three sides of a right triangle (on the same side of the hypotenuse), the sum of the two crescents thus formed is equal to the area of the triangle.

Proof: $\begin{cases} \text{Entire figure} = \frac{1}{8}\pi\overline{AB}^2 + \text{crescent } BDC + \text{crescent } AEC \text{ (?)} \\ \text{Entire figure} = \frac{1}{8}\pi\overline{AC}^2 + \frac{1}{8}\pi\overline{BC}^2 + \triangle ABC \text{ (?)} \end{cases}$

Now use Ax. 1; etc.

20. Show that the theorem of Ex. 19 is true in the case of a right triangle whose legs are 18 and 24.

21. If from any point in a semicircle a line is drawn perpendicular to the diameter and if semicircles are described on the two segments of the original diameter as diameters, the area of the surface bounded by these three semicircles equals the area of a circle whose diameter is the perpendicular first drawn.



Proof: Area $= \frac{1}{2}\pi\left(\frac{a+b}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 =$ etc.

22. Show that the theorem of Ex. 21 is true in the case of a circle with diameter AB equal to 25 and AD equal to 5.

23. If the sides of a circumscribed regular polygon are tangent to the circle at the vertices of an inscribed regular polygon, each vertex of the outer lies on the prolongation of the apothems of the inner polygon, drawn perpendicular to the several sides.

24. The sum of the perpendiculars drawn from any point within a regular n-gon to the several sides is constant [$= n \cdot$ apothem].

25. The area of a circumscribed equilateral triangle is four times the area of the inscribed equilateral triangle.

26. If a point is taken dividing the diameter of a circle into two parts and circles are described upon these parts as diameters, the sum of the circumferences of these two circles equals the circumference of the original circle.

27. Show that the theorem of Ex. 26 is true in the case of a circle the segments of the diameter of which are 7 and 12.

28. The area of an inscribed regular octagon is equal to the product of the diameter by the side of the inscribed square.

29. If squares are described on the six sides of a regular hexagon (externally), the twelve exterior vertices of these squares are the vertices of a regular 12-gon.

30. If the alternate vertices of a regular hexagon are joined by drawing diagonals, another regular hexagon is formed. Also its area is one third of the original hexagon.

31. Show that the theorem of Ex. 18 is true in the case of two concentric circles whose radii are 34 and 16.

32. In the same or equal circles two sectors are to each other as their central angles.

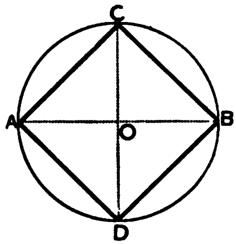
33. If the diameter of a circle is 10 in. and a point is taken dividing the diameter into segments with lengths 4 in. and 6 in., and on these segments as diameters semicircles are described on opposite sides of the diameter, these arcs form a curved line which divides the original circle into two parts in the ratio of 2 : 3.

34. If the diameter of a circle is d and a point is taken dividing the diameter into segments with lengths a and $d - a$, and on these segments as diameters semicircles are described on opposite sides of the diameter, these arcs form a curved line which divides the original circle into two parts in the ratio of $a : d - a$.

CONSTRUCTION PROBLEMS

PROPOSITION XII. PROBLEM

440. To inscribe a square in a given circle.



Given: The circle O.

Required: To inscribe a square.

Construction: Draw any diameter, AB, and another diameter, CD, $\perp$ to AB. Draw AC, BC, BD, AD.

Statement: $ACBD$ is an inscribed square. Q.E.F.

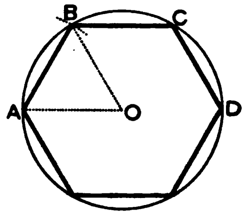
Proof: The central $\angle$s are all equal (42).

$\therefore$ arcs AC, CB, BD, DA are equal (193).

$\therefore$ $ACBD$ is an inscribed square (404). Q.E.D.

PROPOSITION XIII. PROBLEM

441. To inscribe a regular hexagon in a given circle.



Given: (?). **Required:** (?).

Construction: Draw any radius, AO. At A, with radius = to AO, describe arc intersecting the given $\odot$ at B. Draw AB.

Statement: AB is the side of an inscribed regular hexagon.

Proof: Draw BO. $\triangle ABO$ is equilateral (Const.).

$\therefore$ $\triangle ABO$ is equiangular (56).

$\therefore$ $\angle AOB = 60°$ (109).

That is, arc $AB = \frac{1}{6}$ of the circumference; and if arc AB is used as a unit, it divides the circumference into 6 equal arcs. If the chords are drawn, an inscribed regular hexagon is formed (404).

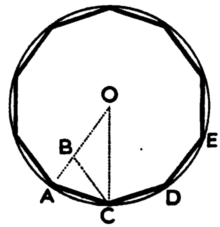
Q.E.D.

Proposition XIV. Problem

442. To inscribe a regular decagon in a given circle.

Given: (?).

Required: (?).



Construction: Draw any radius AO. Divide it into mean and extreme ratio (by 349), having the larger segment next to the center. Taking A as a center and OB as a radius, draw an arc cutting $\odot$ at C. Draw AC, BC, OC.

Statement: AC is a side of the inscribed regular decagon. Q.E.F.

Proof: $AO : BO = BO : AB$ (Const.).

Substituting, $AO : AC = AC : AB$ (Ax. 6).

$\therefore$ △s ABC and AOC are similar (306).

$\therefore$ First, $\angle ACB = \angle O$ (312).

Second, $\triangle ABC$ is isosceles (similar to $\triangle AOC$).

$\therefore AC = BC$ (24).

But $AC = BO$ (Const.).

$\therefore BC = BO$ (Ax. 1).

$\therefore \angle BCO = \angle O$ (55).

Now $\angle ACO = 2\angle O$ (Ax. 4).

$\therefore \angle A = 2\angle O$ (55).

And $\angle O = 1\angle O$

Adding, the ∠s of $\triangle ACO = 5\angle O$ (Ax. 2).

$\therefore 5\angle O = 180°$ (104).

$\therefore \angle O = 36° = \frac{1}{10}$ of $360°$ (Ax. 3).

$\therefore$ arc $AC = \frac{1}{10}$ of the circumference (193).

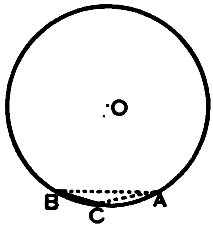
That is, if arc AC is used as a unit it divides the circumference into ten equal arcs; and if the chords are drawn, an inscribed regular decagon is formed. (404).

Q.E.D.

PROPOSITION XV. PROBLEM

443. To inscribe a regular 15-gon (pentadecagon) in a given circle.

Given: (?). **Required:** (?).



Construction: Draw AB, the side of an inscribed hexagon, and AC, the side of an inscribed decagon. Draw BC.

Statement: BC is the side of an inscribed regular 15-gon. Q.E.F.

Proof: Arc BC = arc AB − arc AC

$= \frac{1}{6} - \frac{1}{10}$ of circumference (Const.).

$= \frac{1}{15}$ of circumference.

That is, if arc BC is used as a unit, it divides the circumference into fifteen equal arcs; and if the chords are drawn, an inscribed regular pentadecagon is formed (404). Q.E.D.

444. To inscribe in a given circle:

I. **A regular 8-gon, a regular 16-gon, a regular 32-gon, etc.**

II. **A regular 12-gon, 24-gon, etc.**

III. **A regular 30-gon, 60-gon, etc.**

Construction: I. Inscribe a square. (440.)

Bisect the arcs and draw the chords. **Proof:** (404).

II. Inscribe a regular hexagon, etc.

III. Inscribe a regular pentadecagon, etc.

445. To inscribe an equilateral triangle in a circle.

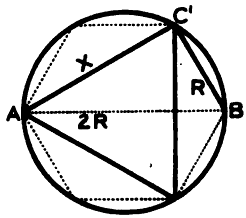
Construction: Join the alternate vertices of an inscribed regular hexagon. **Proof:** (?) (405).

446. To inscribe a regular pentagon in a given circle.

447. To circumscribe a regular polygon about a circle.

Construction: Inscribe a polygon having the same number of sides. At the several vertices draw tangents.

Statement: (?). **Proof:** (?) (406).



FORMULAS

448. Sides of inscribed polygons.

1. **The side of inscribed equilateral triangle** $= R\sqrt{3}$.

Proof: $\angle ACB$ is a rt. $\angle$ (240).

$AB = 2R$ (189).

$BC = R$ (441).

$x^2 = (2R)^2 - R^2$ (335).

$x = R\sqrt{3}$. Q.E.D.

2. **The side of an inscribed square** $= R\sqrt{2}$.

Proof: In fig. of 440

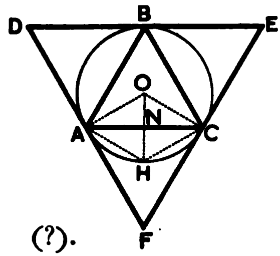
$AO = OC = R$

$\overline{AC}^2 = R^2 + R^2$ (334).

$\therefore AC = R\sqrt{2}$. Q.E.D.

3. **The side of an inscribed regular hexagon** $= R$ (441).

4. **The side of an inscribed regular decagon** $= \frac{1}{2} R(\sqrt{5} - 1)$ (352; 442).



449. Sides of circumscribed polygons.

1. **The side of a circumscribed equilateral** $\triangle = 2R\sqrt{3}$.

Proof: $\angle DAB = \angle DBA = \angle D = 60°$ (?).

$\therefore \triangle ABD$ is equilateral.

$AD = AB = R\sqrt{3}$ (448).

$\therefore DF = 2R\sqrt{3}$. Q.E.D.

2. **The side of a circumscribed square** $= 2R$ (?).

3. **The side of a circumscribed regular hexagon** $= \frac{2}{3} R\sqrt{3}$.

Proof: This side = a side of an equilateral $\triangle$ with altitude R. Let x = this side.

$$\therefore x^2 - (\tfrac{1}{2}x)^2 = R^2 \ (?) \quad \therefore x = \tfrac{2}{3} R\sqrt{3}.$$

450. In an equilateral triangle, **apothem** $= \frac{1}{2} R$.

Proof: Fig. of 449. Bisect arc AC at H.

Draw OA, OC, AH and CH.

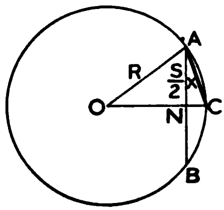
Figure $AOCH$ is a rhombus (448, 3).

$\therefore ON = \frac{1}{2} OH = \frac{1}{2} R$ (135).

Q.E.D.

Proposition XVI. Problem

451. In a circle whose radius is R is inscribed a regular polygon whose side is s; to find the formula for the side of an inscribed regular polygon having double the number of sides.



Given: $AB = s$, a side of an inscribed regular polygon in $\odot$ whose radius is R; C, the midpoint of arc AB; chord AC.

Required: To find the value of AC, the side of a regular polygon having double the number of sides and inscribed in the same circle.

Construction: Draw radii OA and OC.

Computation: OC bisects AB at right $\angle$s (83).

In rt. $\triangle AON$, O is an acute $\angle$. Hence in $\triangle AOC$,

$$\overline{AC}^2 = \overline{OA}^2 + \overline{OC}^2 - 2 \cdot OC \cdot ON \quad (337).$$

But $AO = OC = R$ (187).

And $ON = \sqrt{R^2 - (\frac{1}{2}s)^2} = \frac{1}{2}\sqrt{4R^2 - s^2}$ (335).

Substituting, $\overline{AC}^2 = 2R^2 - 2R \cdot \frac{1}{2}\sqrt{4R^2 - s^2}$ (Ax. 6).

$$\therefore AC = \sqrt{2R^2 - R\sqrt{4R^2 - s^2}}. \quad \text{Q.E.F.}$$

452. Formula. If $R = 1$, and given side $= s$, the side of a regular polygon having twice as many sides $= \sqrt{2 - \sqrt{4 - s^2}}$.

PROPOSITION XVII. PROBLEM

453. To find the approximate numerical value of π.

Given: A circle whose diameter $= D$; circumference $= C$.

Required: The value of π, that is $\frac{C}{D}$.

Method: 1. We may select a ⊙ of *any* diameter. (426.)

2. We can compute the side and the perimeter of some inscribed regular polygon. (448.)

3. We can compute the side and perimeter of another inscribed regular polygon having double the number of sides. (451.)

4. We can now compute the side and perimeter of a third inscribed regular polygon, having still double the number of sides. (451.)

5. By continuing this process until the consecutive perimeters differ very slightly, we can find the approximate value of the circumference.

6. Thus, knowing both C and D, we know $\frac{C}{D}$ or π.

Computation: 1. For simplicity, take $D = 2$, and $R = 1$.

2. We will select the regular hexagon as the first polygon.

$$\therefore S_6 = 1, \text{ and } P_6 = 6.$$

3. $\therefore S_{12} = \sqrt{2 - \sqrt{4 - S_6{}^2}} = \sqrt{2 - \sqrt{3}} = .5176381$ (452).
and $P_{12} = 6.2116572.$

4. Hence $S_{24} = \sqrt{2 - \sqrt{4 - S_{12}{}^2}} = \sqrt{2 - \sqrt{4 - (.5176381)^2}}$
$= .2610524.$

Also $P_{24} = 6.2652576.$

5. By continuing, $S_{3072} = .002045.$

Also $P_{3072} = 6.283184.$

6. It now appears that, approximately, $C = 6.283184.$

Hence $\pi = \frac{6.283184}{2} = 3.141592^{+}.$ Q.E.F.

This calculation is tabulated for reference.

$s_6 = 1,$	$\therefore P_6 = 6.$	$s_{192} = 0.032723,$	$\therefore P_{192} = 6.282904.$
$s_{12} = 0.517638,$	$\therefore P_{12} = 6.211657.$	$s_{384} = 0.016362,$	$\therefore P_{384} = 6.283115.$
$s_{24} = 0.261052,$	$\therefore P_{24} = 6.265257.$	$s_{768} = 0.008181,$	$\therefore P_{768} = 6.283169.$
$s_{48} = 0.130806,$	$\therefore P_{48} = 6.278700.$	$s_{1536} = 0.004091,$	$\therefore P_{1536} = 6.283180.$
$s_{96} = 0.065438,$	$\therefore P_{96} = 6.282053.$	$s_{3072} = 0.002045,$	$\therefore P_{3072} = 6.283184.$

ORIGINAL EXERCISES (NUMERICAL)

Mensuration of Regular Polygons and the Circle

1. Find the angle and the central angle of :
(*i*) a regular pentagon; (*ii*) a regular octagon; (*iii*) a regular dodecagon; (*iv*) a regular 20-gon.

2. Find the area of a regular hexagon whose side is 8.

3. Find the area of a regular hexagon whose apothem is 4.

4. In a circle whose radius is 10 are inscribed an equilateral triangle, a square, and a regular hexagon. Find the perimeter, the apothem, and the area of each.

5. About a circle whose radius is 10 are circumscribed an equilateral triangle, a square, and a regular hexagon. Find the perimeter and the area of each.

6. Find the circumference and the area of a circle whose radius is 5 inches. [Use $\pi = 3\frac{1}{7}$.]

7. Find the circumference and the area of a circle whose diameter is 42 centimeters.

8. The radius of a certain circle is 9 meters. What is the radius of a second circle whose circumference is twice as long as the first? of a third circle whose area is twice as great as the first?

9. If the circumference of a circle is 55 yards, what is its diameter?

10. If the area of a circle is $113\frac{1}{7}$ square meters, what is its radius?

11. In a circle whose radius is 35 there is a sector whose angle is 40°. Find the length of the arc and the area of the sector.

12. The area of a circle is $6\frac{1}{4}$ times the area of another. If the radius of the smaller circle is 12, what is the radius of the larger circle?

13. If the angle of a sector is 72° and its arc is 44 inches, what is the radius of the circle? What is the area of the sector?

14. In a circle whose radius is 7 find the area of the segment whose central angle is 120°. [Required area $= \frac{1}{3}$ (area of $\odot$ − area eq. $\triangle$.]

15. If the radius of a circle is 4 feet, what is the area of a segment whose arc is 60°? of a segment whose arc is a quadrant?

16. Find the area of a circle inscribed in a square whose area is 75.

17. Find the area of an equilateral triangle inscribed in a circle whose area is 441π square meters.

18. If the length of a quadrant is 8 inches, what is the radius?

19. Find the length of an arc subtended by the side of an inscribed regular 15-gon if the radius is $4\frac{2}{3}$ inches.

20. The side of an equilateral triangle is 10. Find the areas of its inscribed and circumscribed circles.

21. Find the perimeter and the area of a segment whose chord is the side of an inscribed regular hexagon, if the radius of a circle is $5\frac{1}{4}$.

22. A circular lake 9 rods in diameter is surrounded by a walk $\frac{1}{2}$ rod wide. What is the area of the walk?

23. A locomotive driving wheel is 7 feet in diameter. How many revolutions will it make in running a mile?

24. What is the number of degrees in the central angle whose arc is as long as the radius?

25. Find the side of the square equal to a circle whose diameter is 4.2 meters.

26. Find the radius of that circle equal to a square whose side is 5.5 inches.

27. Find the radius of the circle which divides a given circle whose radius is $10\frac{1}{2}$ into two equal parts.

28. Three equal circles are each tangent to the other two and the diameter of each is 40 feet. Find the area between these circles.

[Required area = area of an eq. $\triangle$ minus area of three sectors.]

29. Find the area of the three segments of a circle whose radius is $5\sqrt{3}$, formed by the sides of the inscribed equilateral triangle.

30. If a cistern can be emptied in 5 hours by a 2-inch pipe, how long will be required to empty it by a 1-inch pipe?

31. Find the side, the apothem, and the area of a regular decagon inscribed in a circle whose radius is 6 feet.

32. What is the area of the circle circumscribed about an equilateral triangle whose area is $48\sqrt{3}$?

33. The circumferences of two concentric circles are 40 inches and 50 inches. Find the area of the circular ring between them.

34. A circle has an area of 80 square feet. Find the length of an arc of 80°.

35. Find the angle of a sector whose perimeter equals the circumference.

36. Find the angle of a sector whose area is equal to the square of the radius.

37. Find the area of a regular octagon inscribed in a circle whose radius is 20.

[Inscribe square, then octagon. Draw radii of octagon. Find area of one isosceles Δ formed, whose altitude is half the side of the square.]

38. A rectangle whose length is double its width, a square, an equilateral triangle, and a circle all have the same perimeter, namely, 132 meters. Which has the greatest area? the least?

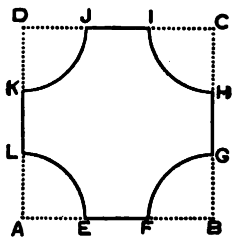
39. Through a point without a circle whose radius is 35 inches two tangents are drawn, forming an angle of 60°. Find the perimeter and the area of the figure bounded by the tangents and their smaller intercepted arc.

40. In a circle whose radius is 12 are two parallel chords which subtend arcs of 60° and 90° respectively. Find the perimeter and the area of the figure bounded by these chords and their intercepted arcs.

41. A quarter mile race track is to be laid out, having parallel sides but semicircular ends with radius 105 feet. Find the length of the parallel sides.

42. If the diameter of the earth is 7920 miles, how far at sea can the light from a lighthouse 150 feet high be seen?

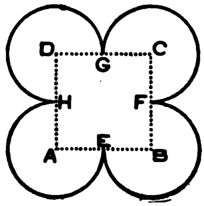
43. The diameter of a circle is 18 inches. Find the area of the figure between this circle and the circumscribed equilateral triangle.



44. How far does the end of the minute hand of a clock move in 20 minutes, if the hand is $3\frac{1}{2}$ inches long?

45. The diameter of a circle is 16 inches. What is the area of that portion of the circle outside the inscribed regular hexagon?

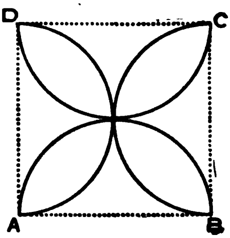
46. Using the vertices of a square whose side is 12, as centers, and radii equal to 4, four quadrants are described within the square. Find the perimeter and the area of the figure thus formed.



47. Using the four vertices of a square whose side is 12 as centers, and radii equal to 6, four arcs are described without the square (see figure). Find the perimeter and the area of the figure bounded by these four arcs.

48. Using the vertices of an equilateral triangle whose side is 16 as centers and radii equal to 8, three arcs are described within the triangle. Find the perimeter and the area of the figure bounded by these arcs. Do the same if the three arcs are described *without* the triangle.

49. Using the vertices of a regular hexagon, whose side is 20, as centers and radii equal to 10, six arcs are described within the hexagon. Find the perimeter and the area of the figure bounded by these arcs. Do the same if the six are described without the hexagon.



50. If semicircumferences are described within a square, with side 8 inches, upon the four sides as diameters, find the areas of the four lobes bounded by the eight quadrants. Find the area of any one.

In the following exercises let n = number of sides of the regular polygon; s = length of side; r = apothem; R = radius; K = area.

51. If $n = 3$, show that $s = R\sqrt{3}$; $r = \frac{1}{2}R$; $K = \dfrac{3R^2\sqrt{3}}{4} = 3r^2\sqrt{3}$.

52. If $n = 4$, show that $s = R\sqrt{2} = 2r$; $K = 2R^2 = 4r^2$.

53. If $n = 6$, show that $s = R = \dfrac{2r\sqrt{3}}{3}$; $K = \dfrac{3R^2\sqrt{3}}{2} = \dfrac{3s^2\sqrt{3}}{2} = 2r^2\sqrt{3}$.

54. If $n = 8$, show that $s = R\sqrt{2-\sqrt{2}} = 2r(\sqrt{2}-1)$; $r = \dfrac{R}{2}\sqrt{2+\sqrt{2}}$;

$$R = \sqrt{4-2\sqrt{2}};\ K = 2R^2\sqrt{2} = 8r^2(\sqrt{2}-1).$$

55. If $n = 10$, show that $s = \dfrac{R}{2}(\sqrt{5}-1)$; $r = \dfrac{R}{4}\sqrt{10+2\sqrt{5}}$.

56. If $n = 5$, show that $s = \dfrac{R}{2}\sqrt{10-2\sqrt{5}}$; $r = \dfrac{R}{4}(\sqrt{5}+1)$.

57. If $n = 12$, show that $s = R\sqrt{2-\sqrt{3}} = 2r(2-\sqrt{3})$;

$$R = 2r\sqrt{2-\sqrt{3}};\ r = \frac{R}{2}\sqrt{2+\sqrt{3}};\ K = 12r^2(2-\sqrt{3}) = 3R^2.$$

58. The apothem of a regular hexagon is $18\sqrt{3}$ inches. Find its side and area. Find the area of the circle circumscribed about it.

59. What is the radius of a circle whose area is doubled by increasing the radius 10 feet?

60. If an 8-inch pipe will fill a cistern in 3 hours 20 minutes, how long will it require a 2-inch pipe to fill it?

61. The radius of a circle is 12 meters. Find:

(*a*) The area of the inscribed square.
(*b*) The area of the inscribed equilateral triangle.
(*c*) The area of the inscribed regular hexagon.
(*d*) The area of the inscribed regular dodecagon.
(*e*) The area of the circumscribed square.
(*f*) The area of the circumscribed equilateral triangle.
(*g*) The area of the circumscribed regular hexagon.
(*h*) The area of the circumscribed regular dodecagon.

62. The radius of a circle is 18. Find:

(*a*) The side and the apothem of the inscribed square.
(*b*) The side and the apothem of the inscribed equilateral triangle.
(*c*) The side and the apothem of the inscribed regular hexagon.
(*d*) The area of the inscribed square.
(*e*) The area of the inscribed equilateral triangle.
(*f*) The area of the inscribed regular hexagon.
(*g*) The area of the inscribed regular octagon.
(*h*) The area of the circumscribed regular hexagon.

63. Prove that the area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and the circumscribed equilateral triangles. [Find the three areas in terms of R.]

64. AB is one side of an inscribed equilateral triangle, and C is the midpoint of AB. If AB is prolonged to O making BO equal to BC, and OT is drawn tangent to the circle at T, OT is $\frac{3}{2}$ the radius.

65. A square, an equilateral triangle, a regular hexagon, and a circle all have the same area, namely 5544 sq. ft. Which figure has the least perimeter? the greatest?

66. A square, an equilateral triangle, a regular hexagon, and a circle all have the same perimeter, namely 396 in. Find their areas and compare them.

67. The circumferences of two concentric circles are 330 and 440 in. respectively. Find the radius of another circle equivalent to the ring between these two circles.

ORIGINAL CONSTRUCTIONS

It is required:

1. To circumscribe a regular hexagon about a given circle.

2. To circumscribe an equilateral triangle about a given circle.

3. To circumscribe a regular decagon about a given circle; a regular 16-gon; a regular 24-gon; a square.

4. To construct an angle of 36°; of 18°; of 72°; of 24°; of 6°; of 48°; of 96°.

5. To construct a regular hexagon upon a given line as a side.

6. To construct a regular decagon upon a given line as a side.

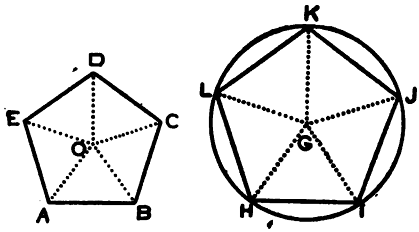
7. To construct a regular octagon upon a given line as a side.

8. To construct a regular pentagon upon a given line as a side.

9. To construct a square with double the area of a given square.

10. To inscribe in a given circle a regular polygon similar to a given regular polygon.

Construction: From the center of the polygon draw radii. At the center of the circle construct ∠s = to these central ∠s of the polygon Draw chords. Etc.



11. To construct a regular pentagon which shall have double the area of a given regular pentagon.

12. To construct a circumference equal to the sum of two given circumferences.

13. To construct a circumference which shall be three times a given circumference.

14. To construct a circumference equal to the difference of two given circumferences.

15. To construct a circle whose area shall be five times a given circle.

16. To construct a circle equal to the sum of two given circles; another, equal to their difference.

17. To construct a circle whose area shall be half a given circle.

18. To bisect the area of a given circle by a concentric circle.

19. To divide a given circumference into two parts which shall be in the ratio of 3 : 7; into two other parts which shall be in the ratio of 5 : 7; into still two other parts, in the ratio of 8 : 7.

MAXIMA AND MINIMA

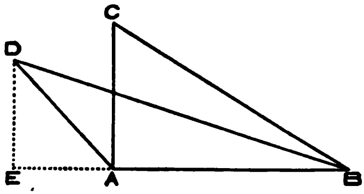
454. Of geometrical magnitudes that satisfy a given condition (or given conditions) the greatest is called the **maximum**, and the least, the **minimum**.

Thus, of all chords that can be drawn through a given point within a circle, the diameter is the maximum, and the chord perpendicular to the diameter at the point is the minimum.

Isoperimetric figures are figures having equal perimeters.

Proposition XVIII. Theorem

455. Of all triangles having two given sides, that in which these sides form a right angle is the maximum.



Given: $\triangle ABC$ and $\triangle ABD$ having AB common, and $AC =$ to AD; $\angle CAB$ a rt. $\angle$ and $\angle DAB$ not a right $\angle$.

To Prove: $\triangle ABC > \triangle ABD$.

Proof: Draw altitude DE.

Now $AD > DE$ (87).

$\therefore AC > DE$ (Ax. 6).

Multiply each member by $\frac{1}{2} AB$.

Then $\frac{1}{2} AB \cdot AC > \frac{1}{2} AB \cdot DE$ (Ax. 10).

Now $\frac{1}{2} AB \cdot AC = \text{area} \triangle ABC$ (364),

And $\frac{1}{2} AB \cdot DE = \text{area } ABD$ (?).

Therefore $\triangle ABC > ABD$ (Ax. 6).

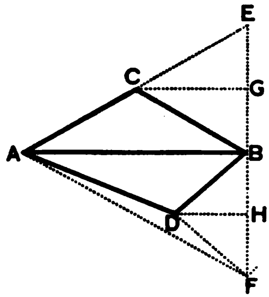
Q.E.D.

This theorem may be stated thus: *Of all triangles having two given sides, that triangle whose third side is the diameter of the circle which circumscribes it is the maximum.*

456. Corollary. **Of all n-gons having $n - 1$ sides given, that polygon whose nth side is the diameter of a circle which circumscribes the polygon is the maximum.**

PROPOSITION XIX. THEOREM

457. Of all isoperimetric triangles having the same base, the isosceles triangle is the maximum.



Given: $\triangle$ *ABC* and *ABD* isoperimetric, having the same base, *AB*, and $\triangle$ *ABC* isosceles.

To Prove: $\triangle ABC > \triangle ABD$.

Proof: Prolong *AC* to *E*, making $CE =$ to *AC*, and draw *BE*. Using *D* as a center and *BD* as a radius, describe an arc cutting *EB* prolonged, at *F*. Draw *CG* and *DH* $\parallel$ to *AB*, meeting *EF* at *G* and *H* respectively. Draw *AF*.

With *C* as a center and *AC*, *BC* or *EC* as a radius, the $\odot$ described will pass through *A*, *B*, and *E* (Hyp. and Const.).

$\therefore \angle ABE =$ rt. $\angle$ (240).

That is, *AB* is $\perp$ to *EF*.

Hence *CG* and *DH* are $\perp$ to *EF* (64).

$AC + CE = AC + CB = AD + DB = AD + DF$ (Hyp. and Const.).

That is, $AE = AD + DF$ (Ax. 1).

But $AD + DF > AF$ (Ax. 12).

$\therefore AE > AF$ (Ax. 6).

$\therefore BE > BF$ (88, IV).

And $\frac{1}{2} BE > \frac{1}{2} BF$ (Ax. 10).

Now $BG = \frac{1}{2} BE$, and $BH = \frac{1}{2} BF$ (85).

$\therefore BG > BH$ (Ax. 6).

Mult. by $\frac{1}{2} AB$, $\frac{1}{2} AB \cdot BG > \frac{1}{2} AB \cdot BH$ (Ax. 10).

But $\frac{1}{2} AB \cdot BG =$ area $\triangle ABC$ (364).

And $\frac{1}{2} AB \cdot BH =$ area $\triangle ABD$ (?).

Substituting, $\triangle ABC > \triangle ABD$ (Ax. 6).

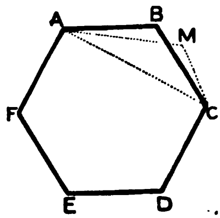
Q.E.D.

458. Corollary. **Of isoperimetric triangles the equilateral triangle is the maximum.**

[Any side may be considered the base.]

Proposition XX. Theorem

459. Of isoperimetric polygons having the same number of sides the maximum is equilateral.



Given: Polygon AD, the maximum of all polygons having the same perimeter and the same number of sides.

To Prove: $AB = BC = CD = DE =$ etc.

Proof: Draw AC and suppose AB not $=$ to BC.

On AC as base, construct $\triangle ACM$ isoperimetric with $\triangle ABC$ *and isosceles;* that is, make $AM = CM$.

Then $\triangle ACM > \triangle ABC$ (457).

Add to each member, the polygon $ACDEF$.

$\therefore$ polygon $AMCDEF >$ polygon AD (Ax. 7).

But the polygon AD is maximum (Hyp.).

$\therefore$ AB cannot be unequal to BC as we supposed (because that results in an impossible conclusion).

Hence $AB = BC$. Likewise it is proved that $BC = CD =$ etc.

Q.E.D.

460. Corollary. **Of isoperimetric polygons having the same number of sides the regular polygon is maximum.**

Proof: Only one such polygon is maximum, and the maximum is equilateral. (459.)

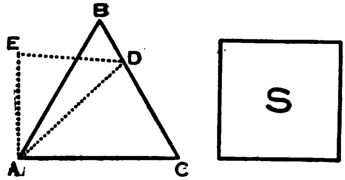
It can also be inscribed in a circle and is therefore regular. (403.)

PROPOSITION XXI. THEOREM

461. Of isoperimetric regular polygons, the polygon having the greatest number of sides is maximum.

Given: Equilateral $\triangle\ ABC$ and square S, having the same perimeter.

To Prove: Square $S > \triangle\ ABC$



Proof: Take D, any point in BC, and draw AD. On AD as base, construct isosceles $\triangle\ ADE$, isoperimetric with $\triangle\ ABD$.

Now $\triangle\ AED > \triangle\ ABD$ (457).

Adding $\triangle\ ADC$ to each member, $AEDC > \triangle\ ABC$ (Ax. 7).

$AEDC$ is isoperimetric with $\triangle\ ABC$ and S (Hyp. and Const.).

Hence $S > AEDC$ (460).

$\therefore\ S > \triangle\ ABC$ (Ax. 11).

Similarly, we may prove that an isoperimetric regular pentagon is greater than S; and an isoperimetric regular hexagon is greater than this pentagon, etc.

Therefore the regular polygon having the greatest number of sides is maximum. Q.E.D.

462. COROLLARY. Of all isoperimetric plane figures the circle is the maximum.

Ex. 1. Of isoperimetric triangles, the maximum is equilateral.

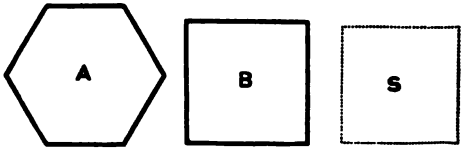
Ex. 2. Of all right triangles that can be constructed upon a given hypotenuse, which is maximum? Why?

Ex. 3. Of all triangles having a given base and a given vertex angle, the isosceles is the maximum.

Ex. 4. Of all mutually equilateral polygons, that which can be inscribed in a circle is the maximum.

PROPOSITION XXII. THEOREM

463. Of equal regular polygons the perimeter of the polygon having the greatest number of sides is the minimum.



Given: Any two equal regular polygons, A and B, A having the greater number of sides.

To Prove: the perimeter of $A <$ the perimeter of B.

Proof: Construct regular polygon S, similar to B and isoperimetric with A.

Then	$A > S$	(460).
But	$A = B$	(Hyp.).
	$\therefore B > S$	(Ax. 6).
Hence	the perimeter of $B >$ perimeter of S	(376).
But	the perimeter of $S =$ perimeter of A	(Const.).
	$\therefore$ perimeter of $B >$ perimeter of A	(Ax. 6).

That is, the perimeter of $A <$ the perimeter of B. Q.E.D.

464. COROLLARY. Of all equal plane figures the circle has the minimum perimeter.

Historical Note. René Descartes was born near Tours, France, in 1596. He was a man of wonderful intellect. He simplified and generalized the notation of algebra and introduced the use of exponents as now employed. The restriction of final letters of the alphabet to represent unknown quantities is also due to him.


DESCARTES

Descartes was the first to adapt algebra to geometry, showing that geometrical figures can be represented by algebraic equations. On this general truth he based the development of analytical geometry which is known by his own name, as *Cartesian* geometry. He gave a large part of his life to original and creative work in mathematics, philosophy, physics and astronomy.

ORIGINAL EXERCISES

1. Of all equal parallelograms having equal bases, the rectangle has the minimum perimeter.

2. Of all lines drawn between two given parallels (terminating both ways in the parallels), which is the minimum? Prove.

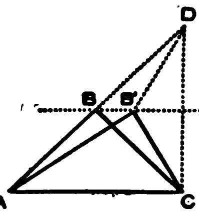
3. Of all straight lines that can be drawn on the ceiling of a room 12 feet long and 9 feet wide, what is the length of the maximum?

4. Find the areas of an equilateral triangle, a square, a regular hexagon, and a circle, the perimeter of each being 264 inches. Which is maximum? What theorem does this exercise illustrate?

5. Find the perimeters of an equilateral triangle, a square, a regular hexagon, and a circle, if the area of each is 1386 square feet. Which perimeter is the minimum? What theorem does this exercise illustrate?

6. Of isoperimetric rectangles which is maximum?

7. Divide a given line into two parts such that their product (rectangle) is maximum.



8. Of all equal triangles having the same base, the isosceles triangle has the minimum perimeter.

To Prove: The perimeter of $\triangle ABC <$ the perimeter of $\triangle AB'C$.

Proof: $AD < AB' + B'D$; etc.

9. Of all rectangles inscribed in a circle, which is maximum? Prove.

10. Of all rectangles inscribed in a semicircle, which is maximum? Prove.

11. Of all equal rectangles, the square has the minimum perimeter.

12. Of all triangles having a given base and a given vertex angle, the isosceles triangle has the maximum area.

13. Of all triangles having a given altitude and a given vertex angle, the isosceles triangle is the minimum.

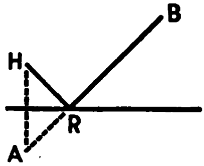
14. Of all triangles that can be inscribed in a given circle, the equilateral triangle has the maximum area.

15. The cross section of a bee's cell is a regular hexagon. Would this be the most economical for the bee (that is, would he use the least wax) if one cell in a hive were all he were to fill? Considering also the adjoining cells, does the form of the regular hexagon require the least wax? Explain. Does it also permit the storing of the most honey? Why?

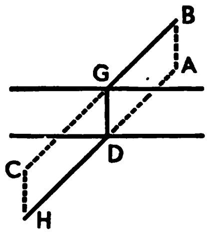
16. Prove, by the method employed in 461, that a regular hexagon is greater than an isoperimetric square.

17. Answer the questions of exercise 65 on page 252, without any computation. Give reasons.

18. Compare the areas of the figures mentioned in Ex. 66, page 252, without performing any computation.



19. A farmer's house and barn are near a river. He wishes to lay from the house to the barn, the shortest possible path which shall reach to the water's edge. Draw a plan of the situation and the desired path and prove it the minimum.



20. A farmer's house and barn are on opposite sides of a straight stream. He wishes to lay a road from one to the other, and erect a bridge across the stream at right angles to the banks, by constructing the shortest possible track. Draw a plan of the situation and the desired road, and prove it the minimum.

21. A man has material to build 1000 yards of fence. With this he desires to inclose the largest possible yard for his poultry. What will be the shape and the area of his yard?

INDEX OF DEFINITIONS

(The numbers refer to pages.)